INSTRUCTIONS: Electronic devices are not permitted during the exam. On the front of your bluebook, write your name, your instructor's name, your lecture number, and a grading table for five problems. Start each problem on a new right-hand page. Justify your work clearly and box your final answer. A correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

1. (25 points) Consider the curve parametrized by $\mathbf{r}(t) = 2\sin(t^2)\,\mathbf{i} + 2\cos(t^2)\,\mathbf{j} + \frac{16}{3}(1+t)^{3/2}\,\mathbf{k}$ for $t \geq 0$.

   (a) Determine the velocity vector.
   SOLUTION: $\mathbf{v}(t) = 4t\cos(t^2)\,\mathbf{i} - 4t\sin(t^2)\,\mathbf{j} + 8(1+t)^{1/2}\,\mathbf{k}$

   (b) Determine the speed along the curve.
   SOLUTION: $|\mathbf{v}|^2 = 16t^2\cos^2(t^2) + 16t^2\sin^2(t^2) + 64(1+t) = 16t^2 + 64t + 64 = 16(t^2+4t+4)$, hence we find the speed to be $|\mathbf{v}| = 4(t+2)$.

   (c) Calculate the arc length $s(t)$ along the curve starting from the point on the curve corresponding to $t = 0$.
   SOLUTION: $\int_0^t 4(\tau+2)\,d\tau = 2t^2 + 8t$.

   (d) Is the velocity vector perpendicular to the position vector when $t = \sqrt{\pi}$?
   SOLUTION: Since $\mathbf{r}(\sqrt{\pi}) = 0\,\mathbf{i} - 2\,\mathbf{j} + \frac{16}{3}(1+\sqrt{\pi})^{3/2}\,\mathbf{k}$ and $\mathbf{v}(\sqrt{\pi}) = -4\sqrt{\pi}\,\mathbf{i} + 0\,\mathbf{j} + 8(1+\sqrt{\pi})^{1/2}\,\mathbf{k}$ we can test to see if $\mathbf{r}(\sqrt{\pi}) \cdot \mathbf{v}(\sqrt{\pi}) = 0$. In fact $\mathbf{r}(\sqrt{\pi}) \cdot \mathbf{v}(\sqrt{\pi}) = \frac{128}{3}(1+\sqrt{\pi})^2 \neq 0$, so no, the velocity and position vectors are not perpendicular when $t = \sqrt{\pi}$.

2. (25 points) Consider two glass plates (planes). The first plane, P_1, intersects the principle axes at the locations $(1,0,0)$, $(0,1,0)$, and $(0,0,2)$. The second plane, P_2, is parallel to the x-axis and intersects the remaining two principle axis at the same points as plane P_1.

   (a) Determine the standard equation of plane P_1 and its unit normal vector $\mathbf{n}_1$.
   SOLUTION: We can form the vectors $\mathbf{P_1P_2} = -1\,\mathbf{i} + 1\,\mathbf{j} + 0\,\mathbf{k}$ and $\mathbf{P_1P_3} = -1\,\mathbf{i} + 0\,\mathbf{j} + 2\,\mathbf{k}$. A (non-unit) normal to plane 1 is then $\mathbf{n}_1 = \mathbf{P_1P_2} \times \mathbf{P_1P_3} = 2\,\mathbf{i} + 2\,\mathbf{j} + 1\,\mathbf{k}$. The unit version would then be $\hat{\mathbf{n}}_1 = \frac{1}{3}(2\,\mathbf{i} + 2\,\mathbf{j} + 1\,\mathbf{k})$. Finally, using the point P_1 and the normal $\mathbf{n}_1$, we can write out the standard equation of plane 1 as $2x + 2y + z = 2$.

   (b) Determine the standard equation of plane P_2 and its unit normal vector $\mathbf{n}_2$.
   SOLUTION: We can form the vectors $\mathbf{P_2P_3} = 0\,\mathbf{i} - 1\,\mathbf{j} + 2\,\mathbf{k}$ and $\mathbf{P_3P_4} = 1\,\mathbf{i} + 0\,\mathbf{j} + 0\,\mathbf{k}$. A (non-unit) normal to plane 1 is then $\mathbf{n}_2 = \mathbf{P_2P_3} \times \mathbf{P_3P_4} = 0\,\mathbf{i} + 2\,\mathbf{j} + 1\,\mathbf{k}$. The unit version would then be $\hat{\mathbf{n}}_2 = \frac{1}{5}(0\,\mathbf{i} + 2\,\mathbf{j} + 1\,\mathbf{k})$. Finally, using the point P_4 and the normal $\mathbf{n}_2$, we can write out the standard equation of plane 2 as $0x + 2y + z = 2$.

   (c) Determine the cosine of the angle between the plates, $\cos\theta$.
   SOLUTION: From the definition of vector dot products, $\cos(\theta) = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{\sqrt{5}}{3}$.

   (d) A laser beam at the origin is aimed perpendicular to plane P_1 and then passes through plane P_2 at point B. What are the coordinates of point B.
   SOLUTION: A line through the origin in the direction of $\mathbf{n}_1$ would be $\mathbf{r}(t) = \mathbf{0} + (t)\mathbf{n}_1 = 2t\,\mathbf{i} + 2t\,\mathbf{j} + t\,\mathbf{k}$. We want the value of t^* that would put a point on the line onto plane 2. Hence we want $0x + 2y + z = 0(2t^*) + 2(2t^*) + (t^*) = 2$, in other words, $t^* = 2/5$. Now, we can calculate $\mathbf{r}(t^*) = 2(2/5)\,\mathbf{i} + 2(2/5)\,\mathbf{j} + (2/5)\,\mathbf{k}$, so B has the coordinates $(4/5, 4/5, 2/5)$.

3. (25 points) Determine the various functions $x(t)$, $y(t)$, and $z(t)$, such that the parameterization of the path $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ satisfies the following unrelated descriptions:

(a) The line through the points $(3, 2, 1)$ and $(6, 5, 4)$.
SOLUTION: We can form the direction vector of the line $\mathbf{D} = (6-3)\,\mathbf{i} + (5-2)\,\mathbf{j} + (4-1)\,\mathbf{k} = 3\,\mathbf{i} + 3\,\mathbf{j} + 3\,\mathbf{k}$. Then $\mathbf{r}(t) = (3\,\mathbf{i} + 2\,\mathbf{j} + 1\,\mathbf{k}) + t\,\mathbf{D} = (3+3t)\,\mathbf{i} + (2+3t)\,\mathbf{j} + (1+3t)\,\mathbf{k}$.

(b) The curve $z = y^2$ in the $x = 2$ plane.
SOLUTION: $\mathbf{r}(t) = 2\,\mathbf{i} + t\,\mathbf{j} + t^2\,\mathbf{k}$ will work.

(c) The curve $x^2 + z^2/9 = 1$ in the $y = 4$ plane.
SOLUTION: $\mathbf{r}(t) = \cos(t)\,\mathbf{i} + 4\,\mathbf{j} + 3\sin(t)\,\mathbf{k}$ will work.

(d) The intersection of the surfaces $z = 2 - x^2 - y^2$ and $z = y^2 - x^2$ for $y > 0$.
SOLUTION: You can set the z's from each equation equal and find that $y = 1$ (don't forget that $y > 0$!) Then from either surface equation, we get $z = 1 - x^2$. Hence $\mathbf{r}(t) = t\,\mathbf{i} + 1\,\mathbf{j} + (1-t^2)\,\mathbf{k}$ will work.

4. (25 points) Consider a particle moving at a constant speed along a curve in space described by $\mathbf{r}(t)$. Although you do not know the form of the function $\mathbf{r}(t)$, you do know that at a given time t^*, $\mathbf{r}(t^*) = 0\,\mathbf{i} + 3\,\mathbf{j} + 4\,\mathbf{k}$, $\mathbf{v}(t^*) = 5\,\mathbf{i} + 5\,\mathbf{j} + 0\,\mathbf{k}$, and $\mathbf{a}(t^*) = 0\,\mathbf{i} + 0\,\mathbf{j} + 7\,\mathbf{k}$. If you can, calculate the following quantities at time $t = t^*$. Otherwise, clearly state that there is insufficient information to perform the calculation. Be sure to explain your reasoning for each calculation!

(a) The unit tangent $\mathbf{T}$.
SOLUTION: Since $|\mathbf{v}| = \sqrt{50}$, then $\mathbf{T}(t^*) = \frac{1}{\sqrt{50}}(5\,\mathbf{i} + 5\,\mathbf{j} + 0\,\mathbf{k}) = \frac{1}{\sqrt{2}}(1\,\mathbf{i} + 1\,\mathbf{j} + 0\,\mathbf{k})$.

(b) The unit normal $\mathbf{N}$.
SOLUTION: Because the speed is constant, the acceleration vector is simply $\mathbf{a} = \kappa|\mathbf{v}|^2\mathbf{N}$ in which case $\mathbf{N} = \mathbf{a}/|\mathbf{a}| = \frac{1}{7}(0\,\mathbf{i} + 0\,\mathbf{j} + 7\,\mathbf{k}) = \mathbf{k}$.

(c) The curvature κ.
SOLUTION: From part (b) above, we see that $|\mathbf{a}| = \kappa|\mathbf{v}|^2$, hence $\kappa = |\mathbf{a}|/|\mathbf{v}|^2 = 7/50$.

(d) The torsion τ.
SOLUTION: There is insufficient information to determine τ because it would require information about the time derivative of $\mathbf{a}$ at time t^*.

(e) Now, assume the time is $t = 2t^*$. What is $\mathbf{v} \cdot \mathbf{a}$? Again, be sure to explain your reasoning here.
SOLUTION: Since the speed is constant, then $\mathbf{v}$ and $\mathbf{a}$ are orthogonal for all t, in partucular at $t = 2t^*$. Thus $\mathbf{v} \cdot \mathbf{a} = 0$ always.

Projections, distances from point S to line containing point P, and S to plane with normal $\mathbf{n}$

$$\text{proj}_{\mathbf{A}}\mathbf{B} = \left(\frac{\mathbf{A}\cdot\mathbf{B}}{\mathbf{A}\cdot\mathbf{A}}\right)\mathbf{A} \qquad d = \frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|} \qquad d = \left|\overrightarrow{PS}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right|$$

Arc length, Frenet formulas, and tangential and normal acceleration components

$$ds = |\mathbf{v}|\,dt \qquad \mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} \qquad \mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \qquad \mathbf{B} = \mathbf{T}\times\mathbf{N}$$

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N} \qquad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N} \qquad \kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{|f''(x)|}{[\,1+(f'(x))^2\,]^{3/2}} = \frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{(\dot{x}^2+\dot{y}^2)^{3/2}} \qquad \tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N}$$

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T} \quad \text{where} \quad a_T = \frac{d|\mathbf{v}|}{dt} \quad \text{and} \quad a_N = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

Fond memories from simpler times

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \qquad \cos^2(x) = \frac{1+\cos(2x)}{2}$$

Fond memories from Calculus II

$$\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+C \qquad \int\sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\ln(x+\sqrt{a^2+x^2})+C$$

$$\int x^2\sqrt{a^2+x^2}\,dx = \frac{x}{8}(a^2+2x^2)\sqrt{a^2+x^2} - \frac{a^4}{8}\ln(x+\sqrt{a^2+x^2})+C$$