1. (20 points) Consider the function \( f(x,y) = x^3 + 3xy^2 + 9y^2 - 75x + 2 \). Find, and if possible, classify all critical points of \( f(x,y) \). If you can’t classify a critical point, write “Indeterminate.”

SOLUTION: Setting \( f_x = 3x^2 + 3y^2 - 75 = 0 \) yields the curve \( x^2 + y^2 = 5^2 \) as the collection of points in the \( x-y \) plane where \( f_x = 0 \). Similarly, setting \( f_y = 6xy + 18y = 0 \) indicates that that the two curves \( x = -3 \) and \( y = 0 \) are the collection of points where \( f_y = 0 \). We need the intersection locations where \( f_x \) and \( f_y \) are simultaneously zero. Hence the critical points are \( CP_1(-3,-4), CP_2(-3,4), CP_3(-5,0) \) and \( CP_4(5,0) \). Evaluating the discriminant \( D = f_{xx}f_{yy} - f_{xy}^2 \) at each of the CPs we see that \( D \) at \( CP_1 \) and \( CP_2 \) is a negative number which indicates that each of these CPs is a saddle point. At \( CP_3 \) the discriminant is positive but \( f_{xx} < 0 \), so \( CP_3 \) is a local max. Finally, at \( CP_4 \) the discriminant is positive and \( f_{xx} > 0 \), so \( CP_4 \) is a local min.

2. (20 points) Consider an open-top box with a square bottom. Let \( s \) be the edge length of the bottom (in meters), and let \( h \) be the height of the box (in meters). The base of the box costs $3 per square meter to construct, while the sides cost only $1 per square meter. Find the dimensions of the box of greatest volume that can be constructed for exactly $36. Be sure to state the resulting box dimensions and box volume.

Note: be sure to use your most advanced Calculus III techniques for this problem to receive full credit!

SOLUTION: All length measurements are in meters. The objective function is \( f(s,h) = Vol(s,h) = s^2h \). The constraint is the \( g(s,h) = Cost(s,h) = 3(s^2) + 1(4sh) = 36 \). Using \( \nabla f = \lambda(\nabla g) \) we get

\[
2sh \mathbf{i} + s^2 \mathbf{j} = \lambda(6s + 4h) \mathbf{i} + \lambda 4s \mathbf{j}
\]

Setting the \( \mathbf{i} \) and \( \mathbf{j} \) components equal, and accounting for the constraint, we get the three coupled algebraic equations

\[
2sh = \lambda(6s + 4h), \quad s^2 = \lambda 4s, \quad 3s^2 + 4sh = 36.
\]

Equating \( \lambda \) from (1) and (2) shows that \( s = 2h/3 \). Using \( s = 2h/3 \) in (3) leads to \( h = 3 \) and \( s = 2 \). Finally, one can calculate the resulting volume to be \( V(2,3) = 12 \).

3. (20 points) Consider the function \( f(x,y,z) = x^2 + y^2 - 3z^2 + z \ln x \).

(a) If \( s \) is arc length, is there a direction \( \mathbf{D} \) in which \( \frac{df}{ds} \) equals 10 at the point \( P_0(1,1,1) \)?

(b) Determine the value of \( \frac{df}{ds} \) as one moves from \( P_0 \) in the direction toward the point \((-1,0,3)\).

(c) Approximate the change in \( f \) as one moves from \( P_0 \) a distance 0.1 toward the point \((-1,0,3)\).
5. (20 points) Consider the function \( f(x,y) = x^3 + y^3 + x^2y^2 + xy \).

(a) Calculate the first order Taylor approximation to \( f(x,y) \) near the point (2,1).

(b) Use your result from part (a) to estimate the value of \( f(2.2, 1.1) \). Do not simplify your answer here. For example, you can leave your answer in the form \( 8 + 4(3.1 - 3) + 3(4.01 - 4) \), although we really do not recommend using these numbers.

(c) Calculate an “upper bound on the error” associated with your first order approximation assuming that you only use values of \( x \) and \( y \) such that \( |x - 2| \leq 0.2 \) and \( |y - 1| \leq 0.1 \). Please simplify your answer, but don’t try to convert to decimal form.

SOLUTION: (a) Since \( f(2,1) = 15 \), \( f_x(2,1) = 17 \), and \( f_y(2,1) = 13 \), we have

\[
 f(x,y) \approx f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) = 15 + 17(x-2) + 13(y-1).
\]

(b) \( f(2.2, 1.1) \approx 15 + 17(0.2) + 13(0.1) \).

(c) We need to search for maximum magnitudes of all second order derivatives (\( f_{xx}, f_{yy}, f_{xy} \)), in a rectangular region bounded by 1.8 \( \leq x \leq 2.2 \) and 0.9 \( \leq y \leq 1.1 \). This results in \( M = f_{yy}(2.2,1.1) = 16.28 \). Hence, the \( |\text{error}| \leq \frac{M}{2!} (|\Delta x| + |\Delta y|)^2 = \frac{f_{yy}(2.2,1.1)}{2} (0.2 + 0.1)^2 = \frac{f_{yy}(2.2,1.1)}{2} (0.09) \).

OVER
Projections and distances
\[
\text{proj}_A B = \left( \frac{A \cdot B}{A \cdot A} \right) A \\
d = \frac{\lVert \vec{P} \times \vec{v} \rVert}{\lVert \vec{v} \rVert} \\
d = \left| \frac{\vec{P} \cdot \vec{n}}{\lVert \vec{n} \rVert} \right|
\]

Arc length, frenet formulas, and tangential and normal acceleration components
\[
ds = \lVert \dot{v} \rVert dt \\
T = \frac{d\vec{r}}{ds} = \frac{\dot{v}}{\lVert \dot{v} \rVert} \\
N = \frac{d\vec{T}/ds}{d\vec{T}/ds} = \frac{d\vec{T}/dt}{d\vec{T}/ds} \\
B = \vec{T} \times \vec{N}
\]
\[
\frac{d\vec{T}}{ds} = \kappa \vec{N} \\
\frac{d\vec{B}}{ds} = -\tau \vec{N} \\
\kappa = \left| \frac{\dot{\vec{v}} \times \vec{a}}{\lVert \dot{\vec{v}} \rVert^3} \right| = \frac{|f''(x)|}{1 + (f'(x))^2}^{3/2} = \frac{|\dot{y} \ddot{x} - \ddot{y} \dot{x}|}{\dot{x}^2 + \dot{y}^2}^{3/2} \\
\tau = -\frac{d\vec{B} \cdot \vec{N}}{ds}
\]
\[
a = a_N \vec{N} + a_T \vec{T} \\
a_T = \frac{d\lVert \dot{v} \rVert}{dt} \\
a_N = \kappa \lVert \dot{v} \rVert^2 = \sqrt{|a|^2 - a_T^2}
\]

Directional derivative, discriminant, and Lagrange multipliers
\[
\frac{df}{ds} = (\nabla f) \cdot \dot{u} \\
f_{xx} f_{yy} - (f_{xy})^2 \\
\nabla f = \lambda \nabla g, \ g = 0
\]

Taylor's formula (at the point \((x_0, y_0)\))
\[
f(x, y) = f(x_0, y_0) + \left[ (x - x_0) f_x(x_0, y_0) + (y - y_0) f_y(x_0, y_0) \right] \\
+ \frac{1}{2!} \left[ (x - x_0)^2 f_{xx}(x_0, y_0) + 2(x - x_0)(y - y_0) f_{xy}(x_0, y_0) + (y - y_0)^2 f_{yy}(x_0, y_0) \right] \\
+ \frac{1}{3!} \left[ (x - x_0)^3 f_{xxx}(x_0, y_0) + 3(x - x_0)^2 (y - y_0) f_{xxy}(x_0, y_0) + 3(x - x_0)(y - y_0)^2 f_{xyy}(x_0, y_0) + (y - y_0)^3 f_{yyy}(x_0, y_0) \right] + \cdots
\]

Linear approximation error
\[
|E(x, y)| \leq \frac{M}{2} (|x - x_0| + |y - y_0|)^2, \quad \text{where } \max\{|f_{xx}|, |f_{xy}|, |f_{yy}|\} \leq M
\]