1. The area of the square is $S^2$ and the area of the triangle is $\sqrt{3}T^2$. Thus, the total area enclosed by the two shapes is $A(S,T) = S^2 + \frac{\sqrt{3}}{4}T^2$. This is the objective function, the function we are trying to maximize. The perimeter of the square is $4S$ and that of the triangle is $3T$. The sum of the perimeters must equal the total length of the wire, $L$. Thus $4S + 3T = L$. The constraint curve can be written as $g(S,T) = 4S + 3T - L = 0$, which, since $S$ and $T$ must both be nonnegative, is a closed, bounded set in $\mathbb{R}^2$. Because $A(S,T)$ is continuous on $\mathbb{R}^2$, it is continuous on the constraint curve, ensuring the existence of a maximum and minimum of $A(S,T)$ on $g(S,T)$. To find the these extrema, use the method of Lagrange multipliers. To this end,

$$\frac{\partial A}{\partial S} = 2S \quad \frac{\partial g}{\partial S} = 4$$

$$\frac{\partial A}{\partial T} = \frac{\sqrt{3}}{2}T \quad \frac{\partial g}{\partial T} = 3$$

leading to the Lagrange equations

$$2S = 4\lambda \implies \lambda = \frac{S}{2}$$

$$\frac{\sqrt{3}}{2}T = 3\lambda \implies \lambda = \frac{\sqrt{3}}{6}T$$

Together, these yield $\frac{S}{2} = \frac{\sqrt{3}}{6}T \implies 3S = \sqrt{3}T \implies T = \sqrt{3}S$. Substituting into the constraint then gives

$$4S + 3\sqrt{3}S - L = 0 \implies \left(4 + 3\sqrt{3}\right) S = L \implies S = \frac{L}{4 + 3\sqrt{3}} \text{ and } T = \frac{\sqrt{3}L}{4 + 3\sqrt{3}}$$

Thus $(S,T) = \left(\frac{L}{4 + 3\sqrt{3}}, \frac{\sqrt{3}L}{4 + 3\sqrt{3}}\right)$ is the only critical point and $A\left(\frac{L}{4 + 3\sqrt{3}}, \frac{\sqrt{3}L}{4 + 3\sqrt{3}}\right) = \frac{L^2}{4 + 3\sqrt{3}}$. Now check the boundaries of the constraint curve, namely, $S = 0$ and $T = 0$.

$$S = 0 \implies T = L/3 \implies A(0,L/3) = \frac{\sqrt{3}L^2}{36} = \frac{L^2}{12\sqrt{3}}$$

$$T = 0 \implies S = L/4 \implies A(L/4,0) = \frac{L^2}{16}$$

Thus, to enclose the maximum area, $S = L/4$ and $T = 0$ (all the wire should be used to create a square).

2. (a) Consider a small portion of $R$ with dimensions $dx \, dy$ centered at $(x,y)$. If this is rotated about the $y$-axis, the volume of the resulting square “tube” is $2\pi y \, dx \, dy$. Thus $I(x,y) = 2\pi y$. 

\[\text{Diagram of a region} \]
(b) Note that \( x \geq 0 \) and \( y \geq 0 \) so only positive square roots need be considered.

\[
\begin{aligned}
  u &= \frac{y^2}{x} \implies u^2 = \frac{y^4}{x^2} \implies y^3 = u^2 v \implies y = \left( u^2 v \right)^{1/3} \implies y(u, v) = u^{2/3} v^{1/3} \\
  v &= \frac{x^2}{y} \implies x^2 = vy = v u^{2/3} v^{1/3} = v^{4/3} u^{2/3} \implies x = \left( v^{4/3} u^{2/3} \right)^{1/2} \implies x(u, v) = u^{1/3} v^{2/3}
\end{aligned}
\]

(c) The region \( \mathcal{R} \) is bounded by the two sets of “parallel curves”, \( y^2 = x, y^2 = 8x \) and \( x^2 = y, x^2 = 8y \) or \( y^2/x = 1, y^2/x = 8 \) and \( x^2/y = 1, x^2/y = 8 \) and can be described as \( \{(x, y) \in \mathbb{R}^2 \mid 1 \leq y^2/x \leq 8, 1 \leq x^2/y \leq 8\} \). Using the given change of variables the new region of integration, \( \mathcal{R}' \), is \( \{(u, v) \in \mathbb{R}^2 \mid 1 \leq u \leq 8, 1 \leq v \leq 8\} \).

(d)

\[
J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix}
\frac{1}{2} u^{-2/3} v^{2/3} & \frac{2}{3} u^{1/3} v^{-1/3} \\
\frac{2}{3} u^{-1/3} v^{1/3} & \frac{1}{3} u^{2/3} v^{-2/3}
\end{vmatrix} = \frac{1}{9} - \frac{4}{9} = -\frac{1}{3}
\]

(e) \( V = \int_1^8 \int_1^8 2\pi u^{2/3} v^{1/3} \left| \frac{1}{3} \right| du dv \)

(f) \( V = \int_1^8 \int_1^8 3 \pi \frac{1}{5} u^{1/3} v^{5/3} \left| \frac{8}{1} \right| dv = \frac{62\pi}{5} \int_1^8 v^{1/3} dv = \frac{62\pi}{5} \frac{8}{4} \left( \frac{3}{1} \right) = \frac{93\pi}{10} (15) = \frac{279\pi}{2} \)

3. (a) The vector field \( \mathbf{F} \) is defined on all of \( \mathbb{R}^2 \), which is open and simply connected. Thus, if \( \nabla \times \mathbf{F} = 0 \), then \( \mathbf{F} \) is conservative.

\[
\nabla \times \mathbf{F} = \begin{vmatrix}
i & j & k \\
\partial/\partial x & \partial/\partial y & \partial/\partial z \\
a (z + xe^y) & (bx^2 + cz) e^y & \sin z + e^y
\end{vmatrix} = (e^y - ce^y) i + a j + (2bx e^y - ax e^y) k = 0,
\]

which will be satisfied if \( a = b = 0 \) and \( c = 1 \), so that \( \mathbf{F} = ze^y \mathbf{j} + (\sin z + e^y) \mathbf{k} \).

(b) Now find the potential function \( \phi(x, y, z) \) for \( \mathbf{F} \) such that \( \mathbf{F} = \nabla \phi \).

\[
\begin{aligned}
\frac{\partial \phi}{\partial x} &= 0 \implies \phi(x, y, z) = g(y, z) \\
\frac{\partial \phi}{\partial y} &= ze^y \implies g(y, z) = \int ze^y \ dy \implies g(y, z) = ze^y + h(z) \implies \phi(x, y, z) = ze^y + h(z) \\
\frac{\partial \phi}{\partial z} &= e^y + \frac{dh}{dz} = \sin z + e^y \implies \frac{dh}{dz} = \sin z \implies h(z) = \int \sin z \ dz \implies h(z) = -\cos z + C
\end{aligned}
\]

so that \( \phi(x, y, z) = ze^y - \cos z + C \) is the sneeze potential function for \( \mathbf{F} \).
(c) Parameterize $C$ as $r(t) = t \mathbf{i} + \pi t \mathbf{k}$, $0 \leq t \leq 1$. Then $F(r(t)) = \pi t \mathbf{j} + (\sin \pi t + 1) \mathbf{k}$, $r'(t) = \mathbf{i} + \pi \mathbf{k}$ and $F(r(t)) \cdot r'(t) = \sin \pi t + 1$. Then

$$\text{Work} = \int_C F \cdot dr = \int_0^1 \pi (\sin \pi t + 1) dt = (-\cos \pi t + \pi t) \bigg|_0^1 = 1 + \pi - (1) = 2 + \pi$$

(d) Since $F$ is conservative, by the Fundamental Theorem for Line Integrals, $\int_C F \cdot dr = \phi(1, 0, \pi) - \phi(0, 0, 0) = \pi e^0 - \cos \pi + C - (0e^0 - \cos 0 + C) = 2 + \pi$

4. (a) With $r(t) = t \mathbf{i} + t^2 \mathbf{j} + (1 - t^2) \mathbf{k}$, $F = -\|\mathbf{v}\| \mathbf{T} = -\|\mathbf{v}\| \frac{\mathbf{v}}{\|\mathbf{v}\|} = -\mathbf{v} = -r'(t) = -2t \mathbf{j} + 2t \mathbf{k}$

$$\text{Work (flow)} = \int_C F \cdot dr = \int_0^1 (-2t \mathbf{j} + 2t \mathbf{k}) \cdot (2t \mathbf{j} - 2t \mathbf{k}) dt = \int_0^1 -8t^2 dt = -\frac{8}{3}$$

(b) Cannot be verified

5. (a) The boundary $S$ of the finite object consists of 3 piecewise smooth surfaces: top $S_1$, side $S_2$ and bottom $S_3$. Consequently the outward flux through the surface is given by

$$\int_S F \cdot dS = \int_{S_1} F \cdot dS + \int_{S_2} F \cdot dS + \int_{S_3} F \cdot dS$$

Surface $S_1$:

$$g(x, y, z) = z, \quad \nabla g = \mathbf{k}, \quad \mathbf{p} = \mathbf{k}, \quad |\nabla g \cdot \mathbf{p}| = 1, \quad F \cdot \frac{\nabla g}{|\nabla g \cdot \mathbf{p}|} = (yi - xj + xyzk) \cdot \mathbf{k} = xyz$$

and with $z = 1$ on $S_1$ this becomes $xy$. The projection of $S_1$ onto the $xy$-plane is the unit circle. Thus

$$\int_{S_1} F \cdot dS = \int_S xy \, dA = \int_0^{2\pi} \int_0^1 r \cos \theta \sin \theta \, r \, dr \, d\theta = \left( \int_0^{2\pi} \frac{1}{2} \sin^2 \theta \, d\theta \right) \left( \int_0^1 r^3 \, dr \right) = 0$$

Surface $S_3$:

$$g(x, y, z) = z, \quad \nabla g = \mathbf{k}, \quad \mathbf{p} = \mathbf{k}, \quad |\nabla g \cdot \mathbf{p}| = 1, \quad F \cdot \frac{\nabla g}{|\nabla g \cdot \mathbf{p}|} = (yi - xj + xyzk) \cdot -\mathbf{k} = -xyz$$

and with $z = 0$ on $S_3$ this becomes 0. The projection of $S_3$ onto the $xy$-plane is the circle of radius $\sqrt{2}$. Note that we used $-\nabla g$ for the outward normal. Thus

$$\int_{S_3} F \cdot dS = \int_S 0 \, dA = 0$$

Surface $S_2$:

$$g(x, y, z) = x^2 + y^2 + z^2, \quad \nabla g = 2xi + 2yj + 2zk, \quad \mathbf{p} = \mathbf{k}, \quad |\nabla g \cdot \mathbf{p}| = |2z| = 2z \text{ since } z \geq 0$$

$$F \cdot \frac{\nabla g}{|\nabla g \cdot \mathbf{p}|} = (yi - xj + xyzk) \cdot \left( \frac{x}{z} \mathbf{i} + \frac{y}{z} \mathbf{j} + \mathbf{k} \right) = xyz$$

and with $z = \sqrt{2 - x^2 - y^2}$ on $S_2$ this becomes $xy\sqrt{2 - x^2 - y^2}$. The projection of $S_2$ onto the $xy$-plane is the annulus with inner radius of 1 and outer radius of $\sqrt{2}$. Thus

$$\int_{S_2} F \cdot dS = \int_S xy \sqrt{2 - x^2 - y^2} \, dA = \int_0^{2\pi} \int_{r=1}^{\sqrt{2}} r \cos \theta \sin \theta \sqrt{2 - r^2} \, r \, dr \, d\theta$$

$$= \left( \int_0^{2\pi} \frac{1}{2} \sin 2\theta \, d\theta \right) \left( \int_{r=1}^{\sqrt{2}} r^3 (2 - r^2) \, dr \right) = 0$$
since the first integral vanishes. Thus

\[ \int \int_S \mathbf{F} \cdot d\mathbf{S} = \int \int_{S_1} \mathbf{F} \cdot d\mathbf{S} + \int \int_{S_2} \mathbf{F} \cdot d\mathbf{S} + \int \int_{S_3} \mathbf{F} \cdot d\mathbf{S} = 0 + 0 + 0 = 0 \]

(b) We can calculate the flux through the boundary \( S \) of the finite object \( W \) using Gauss’ Divergence Theorem, namely

\[ \int \int_S \mathbf{F} \cdot d\mathbf{S} = \iiint_W \nabla \cdot \mathbf{F} \, dV. \]

To this end, \( \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x} (y) + \frac{\partial}{\partial y} (-x) + \frac{\partial}{\partial z} (xyz) = xy \). Spherical coordinates will serve us well here. The region of integration requires two separate integrals, with the following bounds:

\[ 0 \leq \rho \leq \sec \phi \quad 0 \leq \rho \leq \sqrt{2} \]
\[ 0 \leq \phi \leq \pi/4 \quad \pi/4 \leq \phi \leq \pi/2 \]
\[ 0 \leq \theta \leq 2\pi \quad 0 \leq \theta \leq 2\pi \]

The integrand in each integral, including the Jacobian of the transformation, \( \rho^2 \sin \phi \), is

\[ \rho^4 \sin^3 \phi \sin \theta \cos \theta = \frac{1}{2} \rho^4 \sin^3 \phi \sin 2\theta \]

Therefore,

\[ \int \int_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sec \phi} \frac{1}{2} \rho^4 \sin^3 \phi \sin 2\theta \, d\rho \, d\phi \, d\theta + \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{\sqrt{2}} \frac{1}{2} \rho^4 \sin^3 \phi \sin 2\theta \, d\rho \, d\phi \, d\theta \]

\[ = \left( \int_0^{2\pi} \frac{1}{2} \sin 2\theta \, d\theta \right) \left( \int_0^{\pi/4} \int_0^{\sec \phi} \rho^4 \sin^3 \phi \, d\rho \, d\phi \right) + \left( \int_0^{2\pi} \frac{1}{2} \sin 2\theta \, d\theta \right) \left( \int_{\pi/4}^{\pi/2} \int_0^{\sqrt{2}} \rho^4 \sin^3 \phi \, d\rho \, d\phi \right) \]

Noting that the \( \theta \) integrals vanish gives \( \int \int_S \mathbf{F} \cdot d\mathbf{S} = 0 \), verifying the calculation from part (a). Note that cylindrical coordinates could also have been used for the triple integral. In that case, the integral becomes

\[ \int \int_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{2-z^2}} (r \cos \theta) (r \sin \theta) r \, dr \, dz \, d\theta = \left( \int_0^{2\pi} \frac{1}{2} \sin 2\theta \, d\theta \right) \left( \int_0^1 \int_0^{\sqrt{2-z^2}} r^3 \, dr \, dz \right) \]