1. (30 points) You are playing the video game Qbert. Qbert jumps around on the top surfaces of unit cubes (dimensions $1 \times 1 \times 1$) arranged as seen in the pictures below. Assume the center of the surface Qbert stands on, $A$, is located at the origin, that is, the point $A$ is in the $xy$-plane and hence is located at $(0,0,0)$. The coordinate axes are parallel to the edges of the cubes. Qbert can jump from $A$ to the center of adjacent cube surfaces located at points $B$, $C$, $D$, or $E$. To make things more interesting, there are evil coilys in space trying to kill Qbert as it jumps around. When Qbert stands at point $A$, the likelihood of coilys killing it is given by $P(x,y,z) = \exp \left( -\frac{1}{2} ((y-4)^2 + (x-3)^2 - (z-5)^2) \right)$. Be sure to read this function carefully!

(a) Find the coordinates of the points where Qbert can jump, specifically the coordinates of the points $B$, $C$, $D$, and $E$.

(b) If Qbert could pick any direction in space, in what direction should it jump to minimize its chance of being killed?

(c) If Qbert is restricted to jumping to one of the four locations $B$, $C$, $D$, or $E$, where should it jump to minimize its chance of being killed? Warning: don’t simply plug in coordinates, use Calculus 3 concepts to support your answer. Hint: you might find it helpful to note things like $10\sqrt{2} \approx 14$.

2. (30 points) What values of $a$ and $b$ will maximize the value of the integral $\int_{a}^{b} (-3x^2 + 24x - 45) \, dx$? Assume that $a < b$. Be sure to support your answer using Calculus 3 concepts.

3. (30 points) Consider the force field $\mathbf{F} = (y^2 + 2cxz) \mathbf{i} + y(bx + cz) \mathbf{j} + (y^2 + cx^2) \mathbf{k}$.

(a) For what values of $b$ and $c$ will $\mathbf{F}$ be a conservative field?

(b) Using your values for $b$ and $c$, determine the work done by moving on the helical path given by $\mathbf{r}(t) = \cos t \, \mathbf{i} + \sin t \, \mathbf{j} + t \, \mathbf{k}$ from $(1,0,0)$ to $(1,0,2\pi)$.

4. (30 points) Consider the point $(1,1,3)$ in the plane $2x + 2y + z = 7$. Around this point, and in the plane, is a circular path of radius $R$. Calculate the value of the circulation of the vector field $\mathbf{F} = 2y \, \mathbf{i} + 3z \, \mathbf{j} - x \, \mathbf{k}$ around the circular path.

5. (40 points) Consider the path $\mathbf{r}(t)$ formed by the intersection of the two surfaces $z = 2 - x^2 + y^2$ and $z = 1 + 2y^2$, and the vector field given by $\mathbf{F} = -y \, \mathbf{i} + z \, \mathbf{j} + y \, \mathbf{k}$.

(a) Determine a parameterization of the closed path $\mathbf{r}(t)$. Be sure to specify the values of $t$.

(b) Determine the value of the circulation of $\mathbf{F}$ around $\mathbf{r}(t)$ if one moves in a “counter-clockwise” direction.

(c) If it is possible to verify your result in part (b) with another technique, do so. Either way, be sure to justify your reasoning.

6. (40 points) Consider the finite sized object bounded by the surfaces $z = 1 - x^2 - y^2$ and $z = 0$, and the vector field given by $\mathbf{F} = (xz \sin(yz) + x^3) \, \mathbf{i} + \cos(yz) \, \mathbf{j} + (3zy^2 - \exp(x^2 + y^2)) \, \mathbf{k}$. Determine the value of the total outward flux of $\mathbf{F}$ across the bounding surface of the object.
Projections, distances from point $S$ to line containing point $P$, and $S$ to plane with normal $n$

$$\text{proj}_a b = \left(\frac{a \cdot b}{a \cdot a}\right) a$$

$$d = \left\| \frac{\overrightarrow{PS} \times \mathbf{v}}{\|\mathbf{v}\|} \right\|$$

$$d = \left\| \frac{\overrightarrow{PS} \cdot \mathbf{n}}{\|\mathbf{n}\|} \right\|$$

**ARC LENGTH, FRENET FORMULAS, AND TANGENTIAL AND NORMAL ACCELERATION COMPONENTS**

$$ds = \|\mathbf{v}\| \, dt$$

$$\mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

$$\mathbf{N} = \frac{d\mathbf{T}/ds}{\|d\mathbf{T}/ds\|} = \frac{d\mathbf{T}/dt}{\|d\mathbf{T}/dt\|}$$

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$

$$\frac{d\mathbf{T}}{ds} = \kappa \mathbf{N}$$

$$\frac{d\mathbf{B}}{ds} = -\tau \mathbf{N}$$

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\| = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3} = \frac{|f''(x)|}{\left\{1 + [f'(x)]^2\right\}^{3/2}} \quad \tau = -\frac{\mathbf{B} \cdot \mathbf{N}}{\|\mathbf{B}\|}$$

$$\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N} \quad a_T = \frac{d\|\mathbf{v}\|}{dt} \quad a_N = \kappa \|\mathbf{v}\|^2 = \sqrt{\|\mathbf{a}\|^2 - a_T^2}$$

**SECOND DERIVATIVES TEST**

Suppose $f(x, y)$ and its first and second partial derivatives are continuous in a disk centered at $(a, b)$ and $f_x(a, b) = f_y(a, b) = 0$. Let $D = f_{xx}f_{yy} - f_{xy}^2$.

- If $D > 0$ and $f_{xx} < 0$ at $(a, b)$, then $f$ has a local maximum at $(a, b)$.
- If $D > 0$ and $f_{xx} > 0$ at $(a, b)$, then $f$ has a local minimum at $(a, b)$.
- If $D < 0$ at $(a, b)$, then $f$ has a saddle point at $(a, b)$.
- If $D = 0$ at $(a, b)$, then the test is inconclusive.

**DIRECTIONAL DERIVATIVE, DISCRIMINANT, AND LAGRANGE MULTIPLIERS**

$$\frac{df}{ds} = (\nabla f) \cdot \mathbf{u} \quad D = f_{xx}f_{yy} - (f_{xy})^2 \quad \nabla f = \lambda \nabla g, \quad g = 0$$

**TAYLOR’S FORMULA** [at the point $(x_0, y_0)$]

$$f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$+ \frac{1}{2} \left[ f_{xx}(x_0, y_0)(x - x_0)^2 + 2f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + f_{yy}(x_0, y_0)(y - y_0)^2 \right]$$

$$+ \frac{1}{3!} \left[ f_{xxx}(x_0, y_0)(x - x_0)^3 + 3f_{xxy}(x_0, y_0)(x - x_0)^2(y - y_0) \right.$$

$$+ 3f_{xyy}(x_0, y_0)(x - x_0)(y - y_0)^2 + f_{yyy}(x_0, y_0)(y - y_0)^3 \bigg] + \cdots$$

**ERROR IN LINEAR APPROXIMATION**

$$|E(x, y)| \leq \frac{1}{2!} M \left( |x - a| + |y - b| \right)^2, \quad \text{where} \quad \max \{ |f_{xx}|, |f_{xy}|, |f_{yy}| \} \leq M$$

**Coordinate Conversions**

<table>
<thead>
<tr>
<th>Rectangular to Cylindrical</th>
<th>Rectangular to Spherical</th>
<th>Cylindrical to Spherical</th>
</tr>
</thead>
<tbody>
<tr>
<td>(r = \sqrt{x^2 + y^2})</td>
<td>(\rho = \sqrt{x^2 + y^2 + z^2})</td>
<td>(\rho = \sqrt{r^2 + z^2})</td>
</tr>
<tr>
<td>(\theta = \tan^{-1}(y/x))</td>
<td>(\theta = \tan^{-1}(y/r))</td>
<td>(\phi = \tan^{-1}(r/z))</td>
</tr>
<tr>
<td>(z = z)</td>
<td>(\phi = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right))</td>
<td>(\theta = \theta)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Cylindrical to Rectangular</th>
<th>Spherical to Rectangular</th>
<th>Spherical to Cylindrical</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x = r \cos \theta)</td>
<td>(x = \rho \sin \phi \cos \theta)</td>
<td>(r = \rho \sin \phi)</td>
</tr>
<tr>
<td>(y = r \sin \theta)</td>
<td>(y = \rho \sin \phi \sin \theta)</td>
<td>(z = \rho \cos \phi)</td>
</tr>
<tr>
<td>(z = z)</td>
<td>(z = \rho \cos \phi)</td>
<td>(\theta = \theta)</td>
</tr>
</tbody>
</table>
\[ dV = dx \, dy \, dz = r \, dr \, d\theta \, dz = r^2 \sin \phi \, d\rho \, d\phi \, d\theta \]

**CHANGE OF VARIABLES/SUBSTITUTIONS IN MULTIPLE INTEGRALS**

\[
\int \int_{S} f(x, y) \, dA = \int \int_{S} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, du \, dv \quad \text{where} \quad J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}
\]

**MASS, MOMENTS, AND CENTER OF MASS**

Mass \( M = \int \int \delta \, dA \)

Moments \( M_x = \int \int \rho \, dA \quad M_y = \int \int \tau \, dA \quad \text{Center of mass} \quad x = M_y / M \quad y = M_x / M \)

**GREEN’S THEOREM IN THE xy-PLANE**

(The curve \( C \) is traversed counterclockwise and \( \mathbf{F}(x, y) = M(x, y) \, \mathbf{i} + N(x, y) \, \mathbf{j} \))

Circulation \( = \oint_{C} \mathbf{F} \cdot \mathbf{T} \, ds = \int_{R} (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dA = \int_{R} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA \)

Outward Flux \( = \oint_{C} \mathbf{F} \cdot \mathbf{n} \, ds = \int_{R} \nabla \cdot \mathbf{F} \, dA = \int_{R} \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) \, dA \)

**SURFACE AREA OF LEVEL SURFACE** \( g(x, y, z) = c \quad S = \int \int_{S} d\sigma = \int \int_{R} \frac{\| \nabla g \|}{\| \nabla g \cdot \mathbf{p} \|} \, dA \)

**STOKES’ THEOREM** \( \oint_{C} \mathbf{F} \cdot \mathbf{T} \, ds = \int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{S} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_{S} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS \)

**DIVERGENCE/GAUSS’ THEOREM** \( \int_{S} \mathbf{F} \cdot \mathbf{n} \, dS = \int_{S} \mathbf{F} \cdot d\mathbf{S} = \int_{V} \nabla \cdot \mathbf{F} \, dV \)

**FOND MEMORIES FROM BASIC TRIGONOMETRY**

\[
\sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}
\]