1. (30 points) Determine the absolute maximum and minimum values of the function \( f(x, y) = 20 - 16x - 4y + 4x^2 + \frac{y^3}{3} \). Be sure to clearly give both the locations and values of the absolute extremum.

**Solution:**

The given function is a polynomial and therefore its domain is all of \( \mathbb{R}^2 \), which is unbounded. Thus the function need not attain a maximum nor a minimum. Consider the trace for a fixed \( x = x_0 \). Then

\[
\frac{\partial f}{\partial x}(x_0, y) = (20 - 16x_0 - 4x_0^2 - 4y + \frac{y^3}{3})
\]

and

\[
\lim_{y \to \infty} \left[ (20 - 16x_0 - 4x_0^2 - 4y + \frac{y^3}{3}) \right] = \lim_{y \to \infty} \left[ (20 - 16x_0 - 4x_0^2) + y^3 \left(-\frac{4}{y^2} + \frac{1}{3}\right) \right] = \infty
\]

and

\[
\lim_{y \to -\infty} \left[ (20 - 16x_0 - 4x_0^2 - 4y + \frac{y^3}{3}) \right] = \lim_{y \to -\infty} \left[ (20 - 16x_0 - 4x_0^2) + y^3 \left(-\frac{4}{y^2} + \frac{1}{3}\right) \right] = -\infty
\]

implying that \( f(x, y) \) is unbounded and thus does not possess an absolute maximum or minimum.

As an aside, the function may possess local maxima or minima and/or saddle points. Indeed, given \( f(x, y) = 20 - 16x - 4y + 4x^2 + \frac{y^3}{3} \), take the first partial derivatives, set them equal to zero and solve the resulting system of equations.

\[
f_x(x, y) = -16 + 8x = 0 \quad \Rightarrow \quad 8x = 16 \quad \Rightarrow \quad x = 2
\]

\[
f_y(x, y) = -4 + y^2 = 0 \quad \Rightarrow \quad y^2 = 4 \quad \Rightarrow \quad y = \pm 2
\]

Critical points are thus \((2, 2)\) and \((2, -2)\). Now apply the Second Derivatives Test to determine the character of the critical points.

\[
\begin{align*}
D(x, y) &= (8)(2y) - 0^2 = 16y
\end{align*}
\]

Then \( D(2, 2) = 32 > 0 \) and with \( f_{xx}(2, 2) = 8 > 0 \), \((2, 2)\) is a local minimum with value \( f(2, 2) = -\frac{4}{3} \). \( D(2, -2) = -32 < 0 \) so \((2, -2)\) is a saddle with value \( f(2, -2) = \frac{28}{3} \).

2. (30 points) An elliptical lettuce garden is described by the region \( 2x^2 + 2xy + 5y^2 \leq 9 \). The density of lettuce in the garden can be represented by \( l(x, y) = e^{x^2+2y^2} \). Binky the bunny, who lives outside the fence, loves to walk around the perimeter of the fence and nibble on the tasty lettuce. Find the location(s) along the fence where Binky should nibble for the best results, and the location(s) that would be the least productive.

**Solution:**

Use Lagrange multipliers to solve the problem of optimizing the lettuce density \( l(x, y) = e^{x^2+2y^2} \) subject to the constraint \( g(x, y) = 2x^2 + 2xy + 5y^2 = 9 \).
We then need to solve the following system of equations

\[ 2xe^{x^2+2y^2} = (4x + 2y)\lambda \implies \lambda = \frac{xe^{x^2+2y^2}}{2x + y} \tag{1} \]

\[ 4ye^{x^2+2y^2} = (2x + 10y)\lambda \implies \lambda = \frac{2ye^{x^2+2y^2}}{x + 5y} \tag{2} \]

\[ 2x^2 + 2xy + 5y^2 = 9 \tag{3} \]

Equating Eq. (1) and (2) yields

\[ \frac{xe^{x^2+2y^2}}{2x + y} = \frac{2ye^{x^2+2y^2}}{x + 5y} \implies x(x + 5y) = 2y(2x + y) \implies x^2 + 5xy = 4xy + 2y^2 \implies x^2 + xy - 2y^2 = 0 \]

Eq. (3) can be rewritten as \( x^2 + xy = \frac{1}{2}(9 - 5y^2) \) which can be combined with the last equation above to yield

\[ \frac{1}{2} (9 - 5y^2) - 2y^2 = 0 \implies 9 - 5y^2 - 4y^2 = 0 \implies 9 = 9y^2 \implies y = \pm 1 \]

Setting \( y = 1 \) and \( y = -1 \) in Eq. (3) yields

\[ 2x^2 + 2x(1) + 5(1)^2 = 9 \implies 2x^2 + 2x - 4 = 0 \implies x^2 + x - 2 = (x + 2)(x - 1) = 0 \implies x = -2, 1 \]

\[ 2x^2 + 2x(-1) + 5(-1)^2 = 9 \implies 2x^2 - 2x - 4 = 0 \implies x^2 - x - 2 = (x - 2)(x + 1) = 0 \implies x = 2, -1 \]

Consequently the critical points are \((-2, 1), (1, 1), (2, -1), \) and \((-1, -1)\). Evaluating the lettuce density function at these points yields

\[ l(-2, 1) = e^{(-2)^2+2(1)^2} = e^{4+2} = e^6 \]

\[ l(1, 1) = e^{1^2+2(1)^2} = e^{1+2} = e^3 \]

\[ l(2, -1) = e^{2^2+2(-1)^2} = e^{4+2} = e^6 \]

\[ l(-1, -1) = e^{(-1)^2+2(-1)^2} = e^{1+2} = e^3 \]

Therefore, Binky will have the most productive nibbling at \((-2, 1)\) or \((2, -1)\) and the least productive nibbling at \((1, 1)\) and \((-1, -1)\).

3. (30 points) Consider a plate with density (mass per unit area) given by the function \( f(x, y) = x^2 \exp(xy) \).

(a) Evaluate the double integral (which represents the total mass of the plate) \( M_{tot} = \int_0^1 \int_y^1 f(x, y) \, dx \, dy \).

(b) Determine the average value of \( f(x, y) \) over the region of integration.

(c) Set up, but do not evaluate, the integrals to determine the \( x \)-coordinate of the center of mass of the plate.

**Solution:**

(a) The plate is shown in the figure.
Life will be much simpler if we change the order of integration (to avoid integration by parts). Thus

\[
M_{tot} = \int_0^1 \int_y^1 x^2 \exp(xy) \, dx \, dy = \int_0^1 \int_0^x x^2 e^{xy} \, dy \, dx = \int_0^1 x^2 e^{xy} \bigg|_0^x \, dx
\]

\[
= \int_0^1 \left( xe^{x^2} - x \right) \, dx \quad \Rightarrow \quad \frac{1}{2} \int_0^1 e^u \, du - \frac{1}{2} \left[ \frac{1}{2} e^u \right]_0^1 = \frac{1}{2} e^1 - \frac{1}{2} - \frac{1}{2} = \frac{e}{2} - 1
\]

(b) The average value of \( f(x, y) \) is given by \( f_{avg} = \frac{\iiint_{\text{plate}} x^2 e^{xy} \, dA}{\iint_{\text{plate}} dA} \), which is simply the integral from part (a) divided by the area of the plate, which is 1/2. Thus

\[
f_{avg} = \frac{\frac{e}{2} - 1}{\frac{1}{2}} = e - 2
\]

(c)

\[
\bar{x} = \frac{M_y}{M_{tot}} = \frac{\int_0^1 \int_0^x x^3 e^{xy} \, dy \, dx}{\frac{e}{2} - 1}
\]

4. (30 points) Consider the function \( f(x, y) = \exp(xy + y) \).

(a) Verify that \( \nabla f \big|_{(1, 1)} = e^2 \mathbf{i} + 2e^2 \mathbf{j} \).

(b) If one moves from \((1, 1)\) to \((1.1, 1.2)\), estimate the change in the value of \( f \).

(c) If one moves away from \((1, 1)\) in the direction of the vector \( \mathbf{A} = 3 \mathbf{i} + 4 \mathbf{j} \), what is \( df/ds \), where \( s \) is the arc length.

(d) If one moves away from \((1, 1)\) in the direction of \( \mathbf{A} \) at the constant speed \( |V| = v_0 \), what is \( df/dt \)?

(e) If one moves away from \((1, 1)\) along the path, \( y = \left( \frac{1}{4} \right) (x + 1)(3 - x) \), what is \( df/ds \)?

**Solution:**

(a) \( \nabla f(x, y) = ye^{xy + y} \mathbf{i} + (x + 1)e^{xy + y} \mathbf{j} \Rightarrow \nabla f(1, 1) = (1)e^{(1)(1)+1} \mathbf{i} + (1 + 1)e^{(1)(1)+1} \mathbf{j} = e^2 \mathbf{i} + 2e^2 \mathbf{j} \)

(b) We can use the differential here, namely \( \Delta f \approx df = f_x(x, y) \Delta x + f_y(x, y) \Delta y \). In the case here, \( \Delta x = 0.1 \) and \( \Delta y = 0.2 \) and with \( f_x(1, 1) = e^2 \) and \( f_y(1, 1) = 2e^2 \) we have

\[
\Delta f \approx df = e^2(0.1) + 2e^2(0.2) = 0.5e^2
\]
(c) We need a unit vector in the direction of $A$ which, since $\|A\| = \sqrt{3^2 + 4^2} = 5$, is $u = \frac{A}{\|A\|} = \frac{3}{5}i + \frac{4}{5}j$. Then

$$\frac{df}{ds} \bigg|_{(1,1)} = \nabla f(1,1) \cdot u = (e^2 i + 2e^2 j) \cdot \left( \frac{3}{5}i + \frac{4}{5}j \right) = \frac{11}{5}e^2$$

(d) $\frac{df}{dt} = \frac{df}{ds} \frac{ds}{dt} = \frac{11}{5}e^2 v_0$

(e) We can parameterize the path in the usual way since it is a function as $r(t) = t i + \frac{1}{4}(t + 1)(3 - t)j$. Then the direction of the path is given by $r'(t) = i + \left[ \frac{1}{2}(1 - t) \right] j$. The path goes through the point $(1,1)$ when $t = 1$ so $r'(1) = i$, which is already a unit vector. Therefore

$$\frac{df}{ds} = \nabla f(1,1) \cdot r'(1) = (e^2 i + 2e^2 j) \cdot i = e^2$$

5. (40 points) Consider the point $(1,1,3)$ in the plane $2x + 2y + z = 7$. Around this point, and in the plane, is a circular path of radius $R$.

(a) Calculate the value of the circulation of the vector field $F = 2yi + 3zj - xk$ around the circular path.

(b) Calculate the value of the total outward flux of $F$ around the circular path in the plane.

**Solution:**

(a) The circle is not easily parameterized, so we’ll use Stokes’ Theorem to compute the circulation. The surface in this case will be the disk lying in the plane and inside the circle. Call it $S$. We’ll need the curl of $F$, which is

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 3z & -x \end{vmatrix} = -3i + j - 2k$$

as well as the unit normal to the surface, which is $n = \frac{1}{3} (2i + 2j + k)$. Given a counterclockwise traversal of the circle, this is the correct orientation for $n$. Then

$$\text{Circulation} = \int_C F \cdot dr = \iint_S \nabla \times F \cdot n \, dS = \iint_S (-3i + j - 2k) \cdot \frac{1}{3} (2i + 2j + k) \, dS$$

$$= \iint_S -2 \, dS = -2 \iint_S \, dS = -2(\text{Area of } S) = -2\pi R^2$$

(b) We would like to use the flux form of Green’s Theorem, namely

$$\iint_C F \cdot n \, ds = \iint_R \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) \, dA$$

However, since the boundary curve $C$ is not parallel to one of the coordinate planes, the theorem is not applicable. Accepted answers to this problem (not the solution) include mention of the flux/normal form of Green’s theorem and the fact that $\nabla \cdot F = \frac{\partial}{\partial x} (2y) + \frac{\partial}{\partial y} (3z) + \frac{\partial}{\partial z} (-x) = 0$. 

\[\square\]
6. (40 points) Consider the portion of a solid sphere of radius \( R = 1 \) in the first octant. In this object the temperature distribution is given by \( T(x, y, z) = \left( \frac{q}{2} \right) (1 - z^2) \) where \( q \) is a positive constant. You will be working with the vector \( \mathbf{F} = -\kappa \nabla T \) which is called the “heat flux vector”. Here, \( \kappa \), is just another positive, physical constant called the “thermal conductivity”.

(a) Describe the level surfaces of the temperature distribution \( T(x, y, z) \).

(b) Calculate the “heat flux vector” \( \mathbf{F} = -\kappa \nabla T \).

(c) The amount of heat crossing any surface can be found by evaluating the surface integral representing the outward flux of \( \mathbf{F} \) over that surface. Compute the outward flux across each of the four bounding surfaces (bottom, left, back and curved top) of the object. Clearly label each of the four values.

(d) Based on your results in part (c), determine the total amount of heat leaving the object.

(e) Now, verify your answer using an appropriate Calculus 3 theorem. Be sure to state the name of the theorem.

**SOLUTION:**

(a) To determine the level surfaces of \( T(x, y, z) \), we set it equal to a constant, say \( k \). Then

\[
T(x, y, z) = \left( \frac{q}{2} \right) (1 - z^2) = k \implies 1 - z^2 = \frac{2k}{q} \implies z = \pm \sqrt{1 - \frac{2k}{q}},
\]

indicating that the level surfaces are planes parallel to the \( xy \)-plane.

(b) \( \mathbf{F} = -\kappa \nabla T = -\kappa \left( \frac{\partial T}{\partial x} \mathbf{i} + \frac{\partial T}{\partial y} \mathbf{j} + \frac{\partial T}{\partial z} \mathbf{k} \right) = -\kappa \left( 0 \mathbf{i} + 0 \mathbf{j} + -qz \mathbf{k} \right) = \kappa qz \mathbf{k} \)

(c) On the bottom surface of the sphere \( (S_{\text{bottom}}) \), the edge lying the \( xy \)-plane, \( \mathbf{F} = 0 \) (since \( z = 0 \) there) so there is no flux through \( S_{\text{bottom}} \). On the left surface of the sphere \( (S_{\text{left}}) \), the edge lying in the \( xz \)-plane, the outward pointing normal \( \mathbf{n} = -\mathbf{j} \). Thus \( \mathbf{F} \cdot \mathbf{n} = -\kappa qz \mathbf{k} \cdot (-\mathbf{j}) = 0 \) so there is no flux through \( S_{\text{left}} \). Similarly, on the back surface of the sphere \( (S_{\text{back}}) \), the edge lying in the \( yz \)-plane, the outward pointing normal \( \mathbf{n} = -\mathbf{i} \) so that \( \mathbf{F} \cdot \mathbf{n} = -\kappa qz \mathbf{k} \cdot (-\mathbf{i}) = 0 \) and there is no flux through \( S_{\text{back}} \). For the curved top portion of the sphere \( (S_{\text{top}}) \), we have

\[
g(x, y, z) = x^2 + y^2 + z^2 \implies \nabla g = 2x \mathbf{i} + 2y \mathbf{j} + 2z \mathbf{k} \quad (+\nabla g \text{ points upward/outward})
\]

which we will project onto the \( xy \)-plane giving \( \mathbf{p} = \mathbf{k} \), \( |\nabla g \cdot \mathbf{p}| = 2z \) (since \( z \geq 0 \)) and the integration region

\[
\mathcal{R} = \left\{ (x, y) \in \mathbb{R}^2 \left| x \geq 0, y \geq 0, x^2 + y^2 \leq 1 \right. \right\}
\]

Then

\[
\iint_{S_{\text{top}}} \mathbf{F} \cdot \mathbf{dS} = \iint_{\mathcal{R}} \mathbf{F} \cdot \frac{\pm \nabla g}{|\nabla g \cdot \mathbf{p}|} \mathbf{dA} = \iint_{\mathcal{R}} \kappa qz \mathbf{k} \cdot \frac{2x \mathbf{i} + 2y \mathbf{j} + 2z \mathbf{k}}{2z} \mathbf{dA} = \iint_{\mathcal{R}} \kappa qz \mathbf{dA}
\]

This last integral is most efficiently evaluated using polar coordinates, along with \( z = \sqrt{1 - x^2 - y^2} = \sqrt{1 - r^2} \) attained from the surface.

\[
\iint_{\mathcal{R}} \kappa qz \mathbf{dA} = \kappa q \int_{0}^{\pi/2} \int_{0}^{1} r \sqrt{1 - r^2} \, dr \, d\theta \left| u = \frac{1 - r^2}{2} \kappa q \int_{1}^{0} \frac{1}{2} u^{1/2} \, du = -\frac{\pi}{4} \kappa q \frac{2}{3} u^{3/2} \right|_{1}^{0} = \frac{\kappa q \pi}{6}
\]

(d) The total amount of heat leaving the object is

\[
\iint_{S} \mathbf{F} \cdot \mathbf{dS} = \iint_{S_{\text{top}}} \mathbf{F} \cdot \mathbf{dS} + \iint_{S_{\text{left}}} \mathbf{F} \cdot \mathbf{dS} + \iint_{S_{\text{back}}} \mathbf{F} \cdot \mathbf{dS} + \iint_{S_{\text{top}}} \mathbf{F} \cdot \mathbf{dS} = 0 + 0 + 0 + \frac{\kappa q \pi}{6} = \frac{\kappa q \pi}{6}
\]
(e) Gauss’ Divergence theorem is applicable to this portion of space so we can use it to verify our answer in part (c).

First note that \( \nabla \cdot \mathbf{F} = \frac{\partial}{\partial z} (\kappa q z) = \kappa q \). Thus with \( \mathcal{E} \) representing the portion of the solid sphere of radius 1 in the first octant whose boundary is \( S \) we have

\[
\int_S \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{E}} \nabla \cdot \mathbf{F} \, dV = \kappa q \iint_{\mathcal{E}} dV = \kappa q \left( \frac{1}{8} \right) \left( \frac{4\pi}{3} \right) (1)^3 = \frac{\kappa q \pi}{6},
\]

verifying the result obtained in part (d).

\[\blacksquare\]