1. (25 points) An old, dusty red chili processing facility in the south-western portion of New Mexico was built with a floor described by the region in the first quadrant of the $xy$-plane enclosed by the $x$-axis, the line $y = x$, and the arcs $r = 1$ and $r = 2$. The roof of the building has height $z = \sqrt{9 - x^2 - y^2}$. The density of red chili particles in the air (particles per unit volume) is given by $\delta = \sqrt{9 - x^2 - y^2}$.

(a) Calculate the area of the floor. (Use Calculus 3 techniques for the area calculation! Perhaps, polar coordinates might be useful. However, feel free to check your answer with basic geometry.)

(b) Calculate the volume enclosed by the storage building.

(c) Calculate the number of red chili particles in the air in the building.

2. (25 points) Consider the region in the $xy$-plane bounded by the closed curve $x^2 + 9y^2 + 12y + 10x = -25$. Your job is to find the enclosed area, $A$. The substitution $u = 3y + 2$ and $v = x + 5$ will simplify your calculation.

(a) Solve for $x$ and $y$ in terms of $u$ and $v$ using the given substitutions.

(b) Transform the original region $R_{xy}$ in the $x$-$y$ plane into its corresponding region $R_{uv}$ in the $u$-$v$ plane. Make a clear sketch of the new region of integration $R_{uv}$ in the $u$-$v$ plane. Be sure to label all axes, boundaries, intersection points, etc., on your $u$-$v$ plane sketch.

(c) Write the integral for $A$ over the region $R_{uv}$ in the $u$-$v$ plane in terms of $u$ and $v$.

(d) Evaluate $A$ in terms of $u$ and $v$.

3. (25 points) The integrals

\[ V = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/4} \int_{\rho=0}^{2} \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta + \int_{\theta=0}^{2\pi} \int_{\phi=\pi/4}^{\pi/2} \int_{\rho=0}^{1} \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta \]

determine the volume of an object.

(a) Make a clear sketch of the cross-section of the object in the $r$-$z$ plane (this is a constant $\theta$ plane in cylindrical coordinates) clearly labeling the boundaries of the region of integration.

(b) Express $V$ in spherical coordinates using the order $d\phi d\rho d\theta$.

(c) Express $V$ in cylindrical coordinates using the order $dz \, dr \, d\theta$.

(d) Evaluate any of the integrals above (including the original) to determine the value of $V$.

4. (25 points) Consider the path (section 1) starting at the origin along the along the curve $y = x^2$ to the point $(2, 4)$, and then continuing along a straight line (section 2) from the point $(2, 4)$ to the point $(4, 4)$, and the vector field given by $F = 4x \mathbf{i} - 2y \mathbf{j}$.

(a) Sketch the entire path in the $x$-$y$ plane, including the direction of motion.

(b) Clearly give a parametrization for each section of the path (parts 1 and 2), including the limits on $t$.

(c) Calculate the flow along each section of the path $C$. Be sure to clearly state what the flow is for each section.

(d) Calculate the flux along each section of the path $C$. Be sure to clearly state what the flux is for each section.

OVER
Projections and distances \( \proj_{AB} = \left( A \cdot B \right) A \quad \frac{d}{ds} = \frac{\vec{F} \times \vec{v}}{|v|} \quad \frac{d}{ds} = \frac{\vec{F} \cdot \vec{n}}{|n|} \)

Arc length, frenet formulas, and tangential and normal acceleration components

\[
\frac{d\mathbf{T}}{ds} = \kappa \mathbf{N} \quad \mathbf{d} = \mathbf{d} \mathbf{B} = \mathbf{t} \mathbf{N} \quad \kappa = \left| \frac{\mathbf{v} \times \mathbf{a}}{\mathbf{v}|\mathbf{v}|^3} \right| = \frac{|f''(x)|}{1 + (f'(x))^2}^{3/2} - \frac{|x\partial_y - y\partial_x|}{|x^2 + y^2|^{3/2}} \quad \tau = -\frac{d\mathbf{B}}{ds} N
\]

The Second Derivative Test
Suppose \( f(x,y) \) and its first and second partial derivatives are continuous in a disk centered at \((a,b)\) and \( f_x(a,b) = f_y(a,b) = 0 \). Let \( D = f_{xx}f_{yy} - f_{xy}^2 \).

1. If \( D > 0 \) and \( f_{xx} < 0 \) at \((a,b)\), then \( f \) has a local maximum at \((a,b)\).
2. If \( D > 0 \) and \( f_{xx} > 0 \) at \((a,b)\), then \( f \) has a local minimum at \((a,b)\).
3. If \( D < 0 \) at \((a,b)\), then \( f \) has a saddle point at \((a,b)\).
4. If \( D = 0 \) at \((a,b)\), then the test is inconclusive.

Directional derivative, discriminant, and Lagrange multipliers

\[
\frac{df}{ds} = (\nabla f) \cdot \mathbf{u} \quad f_{xx}f_{yy} - (f_{xy})^2 \quad \nabla f = \lambda \nabla g, \quad g = 0
\]

Taylor’s formula (at the point \((x_0, y_0)\))

\[
f(x, y) = f(x_0, y_0) + \left[ (x - x_0)f_x(x_0, y_0) + (y - y_0)f_y(x_0, y_0) \right] + \frac{1}{2!} \left[ (x - x_0)^2 f_{xx}(x_0, y_0) + 2(x - x_0)(y - y_0)f_{xy}(x_0, y_0) + (y - y_0)^2 f_{yy}(x_0, y_0) \right] + \cdots
\]

Linear approximation error

\[
|E(x, y)| \leq \frac{1}{2} M(|x - x_0| + |y - y_0|)^2, \quad \text{where } \max\{|f_{xx}|, |f_{xy}|, |f_{yy}|\} \leq M
\]

Polar coordinates \( x = r \cos \theta \quad y = r \sin \theta \quad r^2 = x^2 + y^2 \quad dA = dx dy = r dr d\theta \)

Cylindrical and spherical coordinates

<table>
<thead>
<tr>
<th>Cylindrical to Rectangular</th>
<th>Spherical to Cylindrical</th>
<th>Spherical to Rectangular</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = r \cos \theta )</td>
<td>( r = \rho \sin \phi )</td>
<td>( x = \rho \sin \phi \cos \theta )</td>
</tr>
<tr>
<td>( y = r \sin \theta )</td>
<td>( z = \rho \cos \phi )</td>
<td>( y = \rho \sin \phi \sin \theta )</td>
</tr>
<tr>
<td>( z = z )</td>
<td>( \theta = \theta )</td>
<td>( z = \rho \cos \phi )</td>
</tr>
</tbody>
</table>

Substitutions in multiple integrals

\[
\int \int_R f(x, y) \, dx \, dy = \int \int_S f(x(u, v), y(u, v)) \, |J(u, v)| \, du \, dv \quad \text{where } J(u, v) = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial u} \frac{\partial y}{\partial v}
\]

Mass, moments, and center of mass

\[
\text{Mass } M = \int \int_R \delta \, dA
\]

\[
\text{Moments } M_x = \int \int_R y \delta \, dA \quad M_y = \int \int_R x \delta \, dA \quad \text{Center of mass } \bar{x} = M_y / M \quad \bar{y} = M_x / M
\]

Flow and flux

\[
\text{Flow } = \int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_C \mathbf{F} \cdot \mathbf{V} \, dt = \int_C \mathbf{F} \cdot \mathbf{dr} = \int_C M \, dx + N \, dy
\]

\[
\text{Flux } = \int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_C M \, dy - N \, dx \quad \text{where } \mathbf{n} = \mathbf{T} \times \mathbf{k}
\]