1. (20 points) Consider the function $f = \sin(xy) + x\sin(y)$, where $x = u^2 + v^2$, and $y = x^3$.

NOTE: Please use partial derivative notation such as either $\frac{\partial g}{\partial w}$ or $g_w$, and fill in specific function details if possible.

(a) Determine $\frac{\partial f}{\partial u}$.

(b) Determine $\frac{\partial f}{\partial v}$.

Note: Be sure to use your most advanced Calculus III techniques for this problem to receive full credit!

2. (20 points) Consider the function $f(x, y) = 9x^2 + y^3 + 3yx^2 - 75y + 2$. Find, and if possible, classify all critical points of $f(x, y)$. If you can’t classify a critical point, write “Indeterminate.”

3. (20 points) Consider the plane $x + 2y + 3z = 1$.

(a) Determine the coordinates on the plane closest to the origin.

(b) Determine the distance from your answer in part (a) to the origin.

Note: Be sure to use your most advanced Calculus III techniques for this problem to receive full credit!

4. (20 points) Consider the function $f(x, y, z) = x^2 + 2y^2 - 3z^2$.

(a) If $s$ is arc length, is there a direction $D$ in which $\frac{df}{ds}$ equals 80 at the point $P_0(1, 1, 1)$? Be sure to explain your reasoning.

(b) Determine the value of $\frac{df}{ds}$ as one moves from $P_0$ in the direction toward the point $(-3, 0, 1)$.

(c) Approximate the change in $f$ as one moves a distance $\Delta s = 0.1$ from $P_0$ toward the point $(-3, 0, 1)$.

5. (20 points) Consider the function $f(x, y) = x^4 + y^4 + x^2y^2$.

(a) Calculate the first order Taylor approximation to $f(x, y)$ around the point $(1, 2)$.

(b) Use your result from part (a) to estimate the value of $f(1.1, 2.2)$. Do not numerically simplify your answer here — you will only hurt yourself and your score. For example, leave your answer in the form $8 + 4(3.1 - 3) + 3(4.01 - 4)$, although we really do not recommend using these numbers.

(c) Calculate an “upper bound on the error” associated with your first order approximation assuming that you only use values of $x$ and $y$ such that $|x - 1| \leq 0.1$ and $|y - 2| \leq 0.2$. Please simplify your answer, but don’t try to convert to decimal form.

OVER
Projections and distances
\[ \text{proj}_A B = \left( \frac{A \cdot B}{A \cdot A} \right) A \]
\[ d = \frac{|\mathbf{P}_S \times \mathbf{v}|}{|\mathbf{v}|} \]
\[ d = \left| \mathbf{P}_S \cdot \frac{n}{|n|} \right| \]

Arc length, frenet formulas, and tangential and normal acceleration components
\[ ds = |\mathbf{v}| \, dt \]
\[ T = \frac{dr}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} \]
\[ N = \frac{dT}{ds} = \frac{dT}{ds} = \frac{dT}{ds} \]
\[ B = T \times N \]
\[ \frac{dT}{ds} = \kappa N \]
\[ \frac{dB}{ds} = -\tau N \]
\[ \kappa = \frac{dT}{ds} = \frac{\mathbf{v} \times \mathbf{a}}{|\mathbf{v}|^3} = \frac{|f''(x)|}{1 + (f'(x))^2} \sqrt{\frac{\hat{y}^2 + \hat{y}^2}{|\mathbf{v}|^2}} = \frac{|\hat{x}^2 - \hat{y}^2|}{|\mathbf{v}|^2} \]
\[ \tau = \frac{dB}{ds} \cdot N \]
\[ a = a_N N + a_T T \]
\[ a_T = \frac{d|\mathbf{v}|}{dt} \]
\[ a_N = \kappa |\mathbf{v}|^2 = \sqrt{|a|^2 - a_T^2} \]

Directional derivative, discriminant, and Lagrange multipliers
\[ \frac{df}{ds} = (\nabla f) \cdot \mathbf{u} \]
\[ f_{xx} f_{yy} - (f_{xy})^2 \]
\[ \nabla f = \lambda \nabla g, \quad g = 0 \]

Taylor’s formula (at the point \((x_0, y_0)\))
\[ f(x, y) = f(x_0, y_0) + \left[ (x - x_0)f_x(x_0, y_0) + (y - y_0)f_y(x_0, y_0) \right] \]
\[ + \frac{1}{2!} \left[ (x - x_0)^2 f_{xx}(x_0, y_0) + 2(x - x_0)(y - y_0)f_{xy}(x_0, y_0) + (y - y_0)^2 f_{yy}(x_0, y_0) \right] \]
\[ + \frac{1}{3!} \left[ (x - x_0)^3 f_{xxx}(x_0, y_0) + 3(x - x_0)^2 (y - y_0)f_{xxy}(x_0, y_0) + 3(x - x_0)(y - y_0)^2 f_{xyy}(x_0, y_0) + (y - y_0)^3 f_{yyy}(x_0, y_0) \right] + \cdots \]

Linear approximation error
\[ |E(x, y)| \leq \frac{M}{2} (|x - x_0| + |y - y_0|)^2, \quad \text{where} \ \max\{|f_{xx}|, |f_{xy}|, |f_{yy}|\} \leq M \]