INSTRUCTIONS: Electronic devices, books, and crib sheets are not permitted. Write your name and your instructor’s name on the front of your bluebook. Work all problems. Show your work clearly. Note that a correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit. Note: this is a Calculus III final exam. While some problems may be solved using techniques from Calculus I or II, you may not receive full credit if you do so, even if the final result is correct.

1. (25 points) What values of $a$ and $b$ will maximize the value of the integral $I(a,b) = \int_{a}^{b} (-2x^2 + 10x - 12) \, dx$? Assume that $a < b$. Be sure to support your answer using Calculus 3 concepts.

2. (25 points) One of your friends had trouble evaluating the double integral $\iint_{R} \exp\left( -\frac{x^2}{4} - \frac{y^2}{9} \right) \, dA$ where $R$ is region inside the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ in the $x$-$y$ plane. You decide to help them out. How should this problem be done?

3. (25 points) Consider the point $(1,1,3)$ in the plane $2x + 2y + z = 7$. Around this point, and in the plane, is a circular path of radius $R$.
   
   (a) Calculate the value of the circulation of the vector field $\mathbf{F} = 2y \mathbf{i} + 3z \mathbf{j} - x \mathbf{k}$ around the circular path.
   
   (b) What is the unit binormal to the path, $\mathbf{B}$?
   
   (c) What is the torsion, $\tau$, for the path?

4. (25 points) Consider the conservative vector field $\mathbf{F} = (1 + yz) \mathbf{i} + (1 + xz) \mathbf{j} + xy \mathbf{k}$. You need to evaluate the path integral $\int_{(1,1,1)}^{(2,2,2)} \mathbf{F} \cdot \mathbf{T} \, ds$.
   
   (a) Show that $\mathbf{F}$ is conservative.
   
   (b) Determine the potential function $f(x, y, z)$.
   
   (c) Evaluate the path integral along the straight line between $(1,1,1)$ and $(2,2,2)$. (Be sure to do a path integral here!)
   
   (d) If it is possible to verify your calculation in part (c), state your reasoning and perform the appropriate calculation.

5. (25 points) The curve $C$ is formed by the intersection of the two surfaces $S_1$ and $S_2$ where $S_1$ is described by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$ and surface $S_2$ is described by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$. The curve of interest, $C$, is an ellipse of finite size with an arc length greater than zero.
   
   (a) Give a parameterization of the curve $C$.
   
   (b) Set up the integral to calculate the arc length of the curve $C$.
   
   (c) Now let $a = b$. Calculate the circulation of the vector field $\mathbf{F} = -y \mathbf{i} + x \mathbf{j} + y \mathbf{k}$ around the curve $C$.
   
   (d) If there is a way to verify your answer, then do so. Otherwise, write “Cannot be verified.” Be sure to explain your reasoning.

6. (25 points) Consider the surface $x^2 + y^2 + z^2 = a^2$ and vector field $\mathbf{F} = 2x \mathbf{i} + 2y \mathbf{j} + 2z \mathbf{k}$.
   
   (a) Evaluate the outward flux of $\mathbf{F}$ over the entire surface using a surface integral.
   
   (b) If possible, use an appropriate Calculus III theorem to check your result from part (a) by performing a different integral calculation. Otherwise, write “Cannot be verified.” Either way, explain your reasoning.

ENJOY YOUR BREAK!
The Divergence Theorem of Gauss

\[ \text{Mass, moments, and center of mass} \]

Polar coordinates

Suppose \( f(x, y) \) and its first and second partial derivatives are continuous in a disk centered at \((a, b)\), and \( f_x(a, b) = f_y(a, b) = 0 \).

Let \( D = f_x f_y - f_x^2 \).

1. If \( D > 0 \) and \( f_x < 0 \) at \((a, b)\), then \( f \) has a local maximum at \((a, b)\).
2. If \( D > 0 \) and \( f_x > 0 \) at \((a, b)\), then \( f \) has a local minimum at \((a, b)\).
3. If \( D < 0 \) at \((a, b)\), then \( f \) has a saddle point at \((a, b)\).
4. If \( D = 0 \) at \((a, b)\), then the test is inconclusive.

Directional derivative, discriminant, and Lagrange multipliers

Taylor’s formula (at the point \((x_0, y_0)\))

\[
f(x, y) = f(x_0, y_0) + \left[(x-x_0)f_x(x_0, y_0) + (y-y_0)f_y(x_0, y_0)\right] + \frac{1}{2!}\left[(x-x_0)^2 f_{xx}(x_0, y_0) + 2(x-x_0)(y-y_0)f_{xy}(x_0, y_0) + (y-y_0)^2 f_{yy}(x_0, y_0)\right] + \cdots
\]

Linear approximation error

\[
|E(x, y)| \leq \frac{M}{2!} \left(|x-x_0| + |y-y_0|\right)^2, \quad \text{where } \max \left\{ |f_{xx}|, |f_{xy}|, |f_{yy}| \right\} \leq M
\]

Polar coordinates

\[
x = r \cos \theta \quad y = r \sin \theta \quad r^2 = x^2 + y^2 \quad dA = dx \, dy = r \, dr \, d\theta
\]

Cylindrical and spherical coordinates

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<td>( x = r \cos \theta )</td>
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\[
dV = dx \, dy \, dz = dz \, r \, dr \, d\theta = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
\]

Substitutions in multiple integrals

\[
\iint_R f(x, y) \, dx \, dy = \int_G \int f(x(u, v), y(u, v)) \left| J(u, v) \right| \, du \, dv \quad \text{where } J(u, v) = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} = \left| \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right|
\]

Mass, moments, and center of mass

\[
\text{Mass } M = \iiint_R \delta \, dA \quad \text{Moments } M_x = \iiint_R y \delta \, dA \quad M_y = \iiint_R x \delta \, dA \quad \text{Center of mass } \bar{x} = M_y/M \quad \bar{y} = M_x/M
\]

Green’s Theorem in the \( x-y \) plane (The curve \( C \) is traversed counterclockwise, and \( F(x, y) = M(x, y) \mathbf{i} + N(x, y) \mathbf{j} \).)

Circulation \[ \int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_0^1 \left[ \int_R (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dA \right] = \int_0^1 \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \, dA \]

Outward Flux \[ \int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_0^1 \int_R \nabla \cdot \mathbf{F} \, dA = \int_0^1 \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) \, dA \]

Surface area of level surface \( g(x, y, z) = c \)

Stokes’ Theorem \[ \oint_C \mathbf{F} \cdot \mathbf{d}r = \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, ds \]

Divergence Theorem of Gauss \[ \int_S \int \int \mathbf{F} \cdot \mathbf{n} \, ds = \int_S \int_D \nabla \cdot \mathbf{F} \, dV \]

Fond memories

\[
sin^2 x = \frac{1 - \cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos 2x}{2}
\]