1. (25 points) A new library is built with a floor described by the region in the first quadrant of the xy-plane enclosed by the x-axis, the line y = x, and the arcs r = 1 and r = 2. The roof of the library has height \( z = \sqrt{4 - x^2 - y^2} \). The library is filled with really old books that give off lots of dust particles when people read them. The density of dust particles in the air (particles per unit volume) is given by \( \delta = \sqrt{4 - x^2 - y^2} \).

(a) Calculate the area of the floor. (Use Calculus III techniques if you want full credit! However, feel free to check your answer with geometry.)

(b) Calculate the volume enclosed by the library.

(c) Calculate the number of dust particles in the library.

2. (25 points) Consider the integral

\[ I = \int \int_{R_{xy}} 8xy \, dx \, dy, \]

where \( R_{xy} \) is the region in the xy-plane bounded by the curves \( x = 0, y = x, y = 1 - x, \) and \( y = 2 - x \).

(a) The substitution \( u = x + y \) and \( v = x - y \) will simplify the evaluation of \( I \). Find \( x \) and \( y \) in terms of \( u \) and \( v \) using the given substitution. Be sure to check this because the rest of the problem depends on this result!

(b) Transform the original region \( R_{xy} \) into its corresponding region \( R_{uv} \) in the uv-plane. Make two clear sketches, one of the original region of integration \( R_{xy} \) in the xy-plane, and one of the new region of integration \( R_{uv} \) in the uv-plane. Be sure to label all axes, boundaries, intersection points, etc., on each sketch if you want full credit.

(c) Rewrite the integral for \( I \) over the region \( R_{uv} \) in the uv-plane in terms of \( u \) and \( v \). Select your order of integration such that your calculation requires just one integral.

(d) Evaluate \( I \) in terms of \( u \) and \( v \).

3. (25 points) The integral

\[ V = \int_{\theta=0}^{2\pi} \int_{r=R}^{\sqrt{3}R} \int_{z=R}^{\sqrt{4R^2-r^2}} r \, dz \, dr \, d\theta \]

calculates the volume, \( V \), of an object.

(a) Make a clear sketch of the cross-section of the object in a rz-plane (this is a constant \( \theta \) plane in cylindrical coordinates) clearly labeling the bounding surfaces of the region of integration. (If you have trouble with this, you may “buy” a sketch of the shape of the region of integration for 5 points. This sketch will only show the shape of the region, so you will still need to supply the remaining details. The offer to buy this sketch ends at 6:00 PM!)

(b) Express \( V \) in cylindrical coordinates using the order \( dr \, dz \, d\theta \).

(c) Express \( V \) in spherical coordinates using the order \( d\rho \, d\phi \, d\theta \).

(d) Express \( V \) in spherical coordinates using the order \( d\phi \, d\rho \, d\theta \).

(e) Evaluate one of the four integrals above to determine the value of \( V \).

4. (25 points) Consider the counter-clockwise path, \( C \), along the boundary of the region inside the curve \( x^2 + y^2 = 4 \), and the vector function given by \( \mathbf{F} = xy \mathbf{i} + xy \mathbf{j} \).

(a) Sketch the path and give a parametrization for path \( C \).

(b) Calculate the total flow (circulation) along path \( C \).

(c) Calculate the total outward flux along path \( C \).

OVER
Projections and distances  \[ \text{proj}_A B = \left( A \cdot \frac{B}{A \cdot A} \right) A \]

\[ d = \frac{|F^2 \times \n|}{|\n|} \quad d = |F^2| \]

Arc length, frenet formulas, and tangential normal acceleration components

\[ ds = |v| dt \quad T = \frac{dr}{ds} = \frac{v}{|v|} \quad N = \frac{dT/ds}{|dT/ds|} \quad B = T \times N \]

\[ \frac{dT}{ds} = \kappa N \quad \frac{dB}{ds} = -\tau N \quad \kappa = \frac{\left| \frac{dv}{dt} \right|}{|v|^3} = \frac{|f''(x)|}{1 + (f'(x))^2}^{3/2} - \left| \frac{\dot{x} \ddot{y} - \ddot{x} \dot{y}}{|x^2 + y^2|^{3/2}} \right| \quad \tau = -\frac{\frac{dB}{ds}}{ds} N \]

The Second Derivative Test

Suppose \( f(x, y) \) and its first and second partial derivatives are continuous in a disk centered at \((a, b)\) and \( f_x(a, b) = f_y(a, b) = 0 \). Let \( D = f_{xx}f_{yy} - f_{xy}^2 \).

1. If \( D > 0 \) and \( f_{xx} < 0 \) at \((a, b)\), then \( f \) has a local maximum at \((a, b)\).
2. If \( D > 0 \) and \( f_{xx} > 0 \) at \((a, b)\), then \( f \) has a local minimum at \((a, b)\).
3. If \( D < 0 \) at \((a, b)\), then \( f \) has a saddle point at \((a, b)\).
4. If \( D = 0 \) at \((a, b)\), then the test is inconclusive.

Directional derivative, discriminant, and Lagrange multipliers

\[ \frac{df}{ds} = (\nabla f) \cdot u \quad f_{xx}f_{yy} - (f_{xy})^2 \quad \nabla f = \lambda \nabla g, \quad g = 0 \]

Taylor’s formula (at the point \((x_0, y_0)\))

\[ f(x, y) = f(x_0, y_0) + \left[ (x - x_0) f_x(x_0, y_0) + (y - y_0) f_y(x_0, y_0) \right] \]

\[ + \frac{1}{2!} \left[ (x - x_0)^2 f_{xx}(x_0, y_0) + 2(x - x_0)(y - y_0) f_{xy}(x_0, y_0) + (y - y_0)^2 f_{yy}(x_0, y_0) \right] \]

\[ + \frac{1}{3!} \left[ (x - x_0)^3 f_{xxx}(x_0, y_0) + 3(x - x_0)^2(y - y_0) f_{xxy}(x_0, y_0) \right. \]

\[ + 3(x - x_0)(y - y_0)^2 f_{xyy}(x_0, y_0) + (y - y_0)^3 f_{yyy}(x_0, y_0) \right] + \cdots \]

Linear approximation error

\[ |E(x, y)| \leq \frac{1}{2} M |(x - x_0) + (y - y_0)|^2, \quad \text{where} \quad \max \{|f_{xx}|, |f_{xy}|, |f_{yy}|\} \leq M \]

Polar coordinates

\[ x = r \cos \theta \quad y = r \sin \theta \quad r^2 = x^2 + y^2 \quad dA = dx \, dy = r \, dr \, d\theta \]

Cylindrical and spherical coordinates

<table>
<thead>
<tr>
<th>Cylindrical to Rectangular</th>
<th>Spherical to Cylindrical</th>
<th>Spherical to Rectangular</th>
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</thead>
<tbody>
<tr>
<td>( x = r \cos \theta )</td>
<td>( r = \rho \sin \phi )</td>
<td>( x = \rho \sin \phi \cos \theta )</td>
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<tr>
<td>( y = r \sin \theta )</td>
<td>( z = \rho \cos \phi )</td>
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<td>( z = z )</td>
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\[ dV = dx \, dy \, dz = r \, dr \, d\theta \, dz = \rho^2 \sin \phi \, dr \, d\phi \, d\theta \]

Substitutions in multiple integrals

\[ \int \int_R f(x, y) \, dx \, dy = \int \int_S f(x(u, v), y(u, v)) \, |J(u, v)| \, du \, dv \quad \text{where} \quad J(u, v) = \frac{\partial (x, y)}{\partial (u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \]

Mass, moments, and center of mass

\[ Mass \quad M = \int \int_R dA \]

\[ \text{Moments} \quad M_x = \int \int_R y \, dA \quad M_y = \int \int_R x \, dA \]

\[ \text{Center of mass} \quad \bar{x} = M_y / M \quad \bar{y} = M_x / M \]

Flow and flux

\[ \text{Flow} = \int_C \mathbf{F} \cdot T \, ds = \int_C \mathbf{F} \cdot \nabla \, dt = \int_C \mathbf{F} \cdot \mathbf{dr} = \int_C M \, dx + N \, dy \]

\[ \text{Flux} = \int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_C M \, dy - N \, dx \]