1. (20 pts) Consider the curve described parametrically by $x(t) = t - 2$ and $y(t) = \ln(\sec t)$ with $0 \leq t < \pi/2$. Find the length of this curve between the points whose $y$-coordinates are $0$ and $\frac{1}{2} \ln 2$.

**Solution:**
Find appropriate values of $t$.

$y = 0 \implies \ln(\sec t) = 0 \implies \sec t = 1 \implies t = 0$

$y = \frac{1}{2} \ln 2 = \ln(\sqrt{2}) = \ln(\sec t) \implies \sec t = \sqrt{2} \implies t = \frac{\pi}{4}$

\[
\frac{dx}{dt} = 1 \quad \text{and} \quad \frac{dy}{dt} = \frac{1}{\sec t} (\sec t)(\tan t) = \tan t \implies \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{1 + \tan^2 t} = \sec t
\]

since $\sec t > 0$ for $0 \leq t \leq \pi/4$. Then

\[
L = \int_{0}^{\pi/4} \sec t \, dt = \ln |\sec t + \tan t|_{0}^{\pi/4} = \ln(1 + \sqrt{2})
\]

2. (20 pts) Let $f(x) = \sec x$.

(a) Approximate $f(x)$ with a second degree Taylor polynomial centered at $a = \pi/4$.

**Solution:**

\[
f(x) = \sec x \implies f(\pi/4) = \sqrt{2}
\]

\[
f'(x) = \sec x \tan x \implies f'(\pi/4) = \sqrt{2}(1) = \sqrt{2}
\]

\[
f''(x) = \sec^3 x + \sec x \tan^2 x \implies f''(\pi/4) = 2\sqrt{2} + \sqrt{2}(1) = 3\sqrt{2}
\]

Thus $\sec x \approx \sqrt{2} + \sqrt{2} \left(x - \frac{\pi}{4}\right) + \frac{3\sqrt{2}}{2!} \left(x - \frac{\pi}{4}\right)^2$.

(b) Use Taylor’s Formula to find an upper bound for the magnitude of the error in using your approximation to $f(x)$ over the range $\frac{\pi}{4} \leq x \leq \frac{\pi}{3}$. You need not simplify your final answer.

**Solution:**
To apply Taylor’s formula we need $f'''(x) = 5 \sec^3 x \tan x + \sec x \tan^3 x$. Since $\sec x$ and $\tan x$ are both increasing on $\frac{\pi}{4} \leq x \leq \frac{\pi}{3}$, $f'''(x)$ is increasing there as well, implying that its maximum value will occur at the right end point of the interval, $\pi/3$, with a value of

\[
f'''(\pi/3) = 5 \sec^3(\pi/3) \tan(\pi/3) + \sec(\pi/3) \tan^3(\pi/3) = 5(2^3)3\sqrt{3} + 2(3\sqrt{3}) = 46\sqrt{3}
\]

Then, since $x - \frac{\pi}{4}$ is maximized at $\frac{\pi}{3}$,

\[
|R_2(x)| = \left|\frac{f'''(z)}{3!} \left(x - \frac{\pi}{4}\right)^3\right|, \quad \frac{\pi}{4} < z < \frac{\pi}{3} \implies |R_2(x)| \leq \frac{46\sqrt{3}}{6} \left(\frac{\pi}{12}\right)^3 = \frac{23\pi^3\sqrt{3}}{5184}
\]

3. (10 pts) Find an antiderivative of $\frac{\sqrt{x^2 - 25}}{x}$.
\textbf{Solution:}

We need to compute \( \int \frac{\sqrt{x^2 - 25}}{x} \, dx \) so we will use the substitution \( x = 5 \sec \theta \). Then \( dx = 5 \sec \theta \tan \theta \, d\theta \) and

\[
\int \frac{\sqrt{x^2 - 25}}{x} \, dx = \int \frac{\sqrt{25 \sec^2 \theta - 25}}{5 \sec \theta} (5 \sec \theta \tan \theta) \, d\theta = 5 \int \sqrt{\sec^2 \theta - 1} \tan \theta \, d\theta = 5 \int \tan^2 \theta \, d\theta
\]

From the right triangle built using \( \sec \theta = \frac{x}{5} \) we have \( \tan \theta = \frac{\sqrt{x^2 - 25}}{5} \) and \( \theta = \sec^{-1} \left( \frac{x}{5} \right) \) so that

\[
\int \frac{\sqrt{x^2 - 25}}{x} \, dx = \sqrt{x^2 - 25} - 5 \sec^{-1} \left( \frac{x}{5} \right) + C
\]

4. (20 pts) Consider the equations \( r = 4 \sin \theta \) and \( r = 2 \).

(a) Graph these functions on a common polar graph.

\textbf{Solution:}

\begin{center}
\begin{tikzpicture}
\end{tikzpicture}
\end{center}

(b) Find the area of the common interior of the two graphs.

\textbf{Solution:}

The graphs intersect when \( 4 \sin \theta = 2 \implies \theta = \frac{\pi}{6}, \frac{5\pi}{6} \). We’ll exploit the symmetry, finding the area of the region in the first quadrant and doubling the result.

\[
A = 2 \left[ \frac{1}{2} \int_0^{\pi/6} (4 \sin \theta)^2 \, d\theta + \frac{1}{2} \int_{\pi/6}^{\pi/2} 2^2 \, d\theta \right] = \int_0^{\pi/6} 16 \left( \frac{1 - \cos 2\theta}{2} \right) \, d\theta + 4 \int_{\pi/6}^{\pi/2} \, d\theta
\]

\[
= 8 \left( \theta - \frac{1}{2} \sin 2\theta \right) \bigg|_0^{\pi/6} + 4\theta \bigg|_{\pi/6}^{\pi/2} = 8 \left( \frac{\pi}{6} - \frac{\sqrt{3}}{4} \right) + 4 \pi = \frac{8\pi}{3} - 2\sqrt{3}
\]

5. (30 pts) Determine if each of the following converges or diverges. Justify your answers.

(a) \( \left\{ \frac{\tan^{-1} n}{\sqrt{n}} \right\} \)

(b) \( \sum_{n=1}^{\infty} \frac{\tan^{-1} n}{\sqrt{n}} \)

(c) \( \frac{\cos(2\pi)}{(\ln 2)^2} + \frac{\cos(3\pi)}{(\ln 3)^2} + \frac{\cos(4\pi)}{(\ln 4)^2} + \cdots \)

\textbf{Solution:}
(a) For \( n \geq 1 \), we have \( 0 \leq \tan^{-1} n \leq \pi/2 \) \( \implies \) \( 0 \leq \frac{\tan^{-1} n}{\sqrt{n}} \leq \frac{\pi/2}{\sqrt{n}} \).

Since \( \lim_{n \to \infty} \frac{\pi/2}{\sqrt{n}} = \lim_{n \to \infty} 0 = 0, \lim_{n \to \infty} \frac{\tan^{-1} n}{\sqrt{n}} = 0 \) by the Squeeze Theorem. Thus the sequence converges.

(b) Consider \( \lim_{n \to \infty} \frac{\tan^{-1} n}{\sqrt{n}} = \lim_{n \to \infty} \tan^{-1} n = \frac{\pi}{2} \) which is positive and finite. Since \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \) is a \( p \)-series with \( p = 1/2 < 1 \), it diverges and thus \( \sum_{n=1}^{\infty} \frac{\tan^{-1} n}{\sqrt{n}} \) diverges by the limit comparison test.

(c) This is an alternating series \( \left( \sum_{n=2}^{\infty} \frac{\cos(n\pi)}{(\ln n)^2} \right) \) with \( b_n = \frac{1}{(\ln n)^2} \), \( \lim_{n \to \infty} \frac{1}{(\ln n)^2} = 0 \) and \( n + 1 > n \) \( \implies \)

\[ \ln(n + 1) > \ln(n) \implies [\ln(n + 1)]^2 > [\ln(n)]^2 \implies b_n = \frac{1}{[\ln(n)]^2} > \frac{1}{[\ln(n + 1)]^2} = b_{n+1} \] Thus by the Alternating Series Test, \( \sum_{n=2}^{\infty} \frac{\cos(n\pi)}{(\ln n)^2} \) converges.

6. (20 pts) Evaluate \( \int_{0}^{1} x \ln x^3 \, dx \).

**SOLUTION:**

\[
\int_{0}^{1} x \ln x^3 \, dx = \lim_{t \to 0^+} \int_{t}^{1} 3x \ln x \, dx = \int_{t}^{1} 3x \, dx = \frac{3}{2} x^2 \bigg|_{t}^{1} = \frac{3}{2} (1 - t^2)
\]

\[= 3 \lim_{t \to 0^+} \left( \frac{1}{2} x^2 \ln x \right)_{t}^{1} - \int_{t}^{1} \frac{1}{2} x^2 \, dx = 3 \lim_{t \to 0^+} \left( \frac{1}{2} t^2 \ln t - \frac{1}{4} t^2 \right)_{t}^{1} = 3 \lim_{t \to 0^+} \left( \frac{1}{2} t^2 \ln t - \frac{1}{4} + \frac{1}{4} \right) = 3 \lim_{t \to 0^+} \left( \frac{1}{4} t^2 \right) - \frac{1}{4} = -\frac{3}{4}
\]

7. (20 pts) Find the volume of the solid obtained by rotating the region bounded by \( y = x^2 + 1 \) and \( y = 9 - x^2 \) about \( y = -1 \).

**SOLUTION:**

Need points of intersection of the two graphs: \( x^2 + 1 = 9 - x^2 \implies 2x^2 = 8 \implies x = \pm 2 \).

Disks

\[ V = \pi \int_{-2}^{2} \left\{ [(9 - x^2) - (-1)]^2 - [(x^2 + 1) - (-1)]^2 \right\} \, dx \]

\[ = \pi \int_{-2}^{2} \left[ 100 - 20x^2 + x^4 - (x^4 + 4x^2 + 4) \right] \, dx \]

\[ = \pi \int_{-2}^{2} (96 - 24x^2) \, dx = 24(2)\pi \int_{0}^{2} (4 - x^2) \, dx = 48\pi \left( 4x - \frac{1}{3} x^3 \right)_{0}^{2} = 256\pi \]
Shells

\[ V = 2\pi \int_1^5 (y + 1) \left( \sqrt{y - 1} - \left( -\sqrt{y - 1} \right) \right) \, dy + 2\pi \int_5^9 (y + 1) \left( \sqrt{9 - y} - \left( \sqrt{9 - y} \right) \right) \, dy \]

\[ = 4\pi \int_1^5 (y + 1) \sqrt{y - 1} \, dy + 4\pi \int_5^9 (y + 1) \sqrt{9 - y} \, dy \quad u = y - 1 \text{ and } v = 9 - y \]

\[ = 4\pi \int_0^4 (u + 2)\sqrt{u} \, du + 4\pi \int_0^4 (10 - v)\sqrt{v} \, dv \]

\[ = 4\pi \int_0^4 \left( u^{3/2} + 2u^{1/2} \right) \, du + 4\pi \int_0^4 \left( 10v^{1/2} - v^{3/2} \right) \, dv = 256\pi \]

8. (10 pts) Sketch the graph of \( 9x^2 - 18x + 4y^2 = 27 \), properly labeling the vertices.

**SOLUTION:**

\[ 9(x^2 - 2x + 1 - 1) + 4y^2 = 27 \implies 9(x - 1)^2 + 4y^2 = 36 \implies \left( \frac{x - 1}{2} \right)^2 + \left( \frac{y}{3} \right)^2 = 1 \]
Frequently Used Maclaurin Series

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad R = 1
\]

\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad R = \infty
\]

\[
\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad R = \infty
\]

\[
\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad R = \infty
\]

\[
\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)} \quad R = 1
\]

\[
\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \quad R = 1
\]

\[
(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \quad R = 1
\]

Taylor’s Formula

\[
R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} (x-a)^{n+1}
\]

Center of Mass/Centroid Integrals

\[
m = \int_a^b \rho [f(x) - g(x)] \, dx
\]

\[
M_x = \int_a^b \frac{1}{2} \rho \left\{ [f(x)]^2 - [g(x)]^2 \right\} \, dx
\]

\[
M_y = \int_a^b \rho x [f(x) - g(x)] \, dx
\]

\[
\bar{x} = \frac{M_y}{m}, \quad \bar{y} = \frac{M_x}{m}
\]

Midpoint Rule

\[
M_n = \Delta x \left[ f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n) \right] \text{ where } \Delta x = \frac{b-a}{n} \text{ and } \bar{x}_i = \frac{1}{2} (x_{i-1} + x_i)
\]

\[
|E_M| \leq \frac{K(b-a)^3}{24n^2}
\]

Trapezoidal Rule

\[
T_n = \frac{\Delta x}{2} \left[ f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n) \right] \text{ where } \Delta x = \frac{b-a}{n}
\]

\[
|E_T| \leq \frac{K(b-a)^3}{12n^2}
\]