1. (35 points) Determine whether the series is absolutely convergent, conditionally convergent, or divergent. Show all work, and state any theorems or tests that you use.

(a) \[ \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}+4} \]

(b) \[ \sum_{n=2}^{\infty} \frac{1}{n(2+\ln n)^{3/2}} \]

(c) \[ \sum_{n=1}^{\infty} \left( \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+2}} \right) \]

(d) \[ \sum_{n=1}^{\infty} (-1)^n \frac{9}{n^{5/3}} \]

(e) \[ \sum_{n=3}^{\infty} \frac{(-1)^n + \cos n}{n^{7/5}} \]

Solution:

(a) Let \( \sum a_n = \sum \frac{1}{\sqrt{n}+4} \) and \( \sum b_n = \sum \frac{1}{n^{1/2}} \). Then \( \frac{a_n}{b_n} = \frac{\sqrt{n}}{\sqrt{n+4}} = \frac{1}{1+\frac{4}{\sqrt{n}}} \to 1 \). Since \( \sum b_n \) is a divergent \( p \)-series, we conclude by the LCT that \( \sum a_n \) is also divergent.

(b) The function \( f(x) = \frac{1}{x(2+lnx)^{3/2}} \) is continuous, positive, and decreasing for \( x \geq 2 \).

\[ \int_2^t \frac{1}{x(2+\ln x)^{3/2}} \, dx = \left[ \frac{-2}{\sqrt{2+\ln x}} \right]_2^t \to \frac{2}{\sqrt{2+\ln 2}} \text{ as } t \to \infty. \]

Since the corresponding improper integral converges, the series must also converge by the Integral Test. Since the terms are positive, the series is absolutely convergent.

(c)

\[ S_N = \sum_{n=1}^{N} \left( \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+2}} \right) \]

\[ = \left( \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{3}} \right) + \left( \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{4}} \right) + \cdots + \left( \frac{1}{\sqrt{N-1}} - \frac{1}{\sqrt{N+1}} \right) + \left( \frac{1}{\sqrt{N}} - \frac{1}{\sqrt{N+2}} \right) \]

\[ = 1 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{N+1}} - \frac{1}{\sqrt{N+2}} \to 2 + \frac{\sqrt{2}}{2} \text{ as } N \to \infty. \]

Since the terms are all positive, the series is absolutely convergent.

(d)

\[ b_n = \frac{9}{n^{2/3}} > 0. \]

\[ b_{n+1} = \frac{9}{(n+1)^{2/3}} < \frac{9}{n^{2/3}} = b_n. \]

\[ b_n = \frac{9}{n^{2/3}} \to 0 \text{ as } n \to \infty. \]
By the Alternating Series Test, the series is convergent. However,

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{9}{n^{2/3}} > \sum_{n=1}^{\infty} \frac{1}{n^{2/3}},$$

so $\sum |a_n|$ is divergent by the Comparison Test with the divergent $p$-series $\sum \frac{1}{n^{2/3}}$. We conclude that the series is conditionally convergent.

(e)

$$\sum_{n=3}^{\infty} \left| \frac{(-1)^n + \cos n}{n^{7/5}} \right| \leq \sum_{n=3}^{\infty} \frac{2}{n^{7/5}} = 2 \sum_{n=3}^{\infty} \frac{1}{n^{7/5}},$$

which is a convergent $p$-series multiplied by 2.

By the Direct Comparison Test the series is absolutely convergent.

2. (22 points) The following parts are unrelated.

(a) Find the Maclaurin series of the function $f(x) = \frac{3}{(x - 2)^2}$. What is the radius of convergence of the series? Simplify your answers.

(b) Use a Maclaurin series to evaluate $\lim_{x \to 0} \left( \frac{\cos x - 1 - x^2}{5x^2} \right)$

Solution:

(a) 

$$\int f(x) \, dx = -3(x - 2)^{-1} + C = \frac{3}{2} \left( \frac{1}{1 - \frac{x}{2}} \right) + C = \frac{3}{2} \sum_{n=0}^{\infty} \left( \frac{x}{2} \right)^n + C,$$

$$\therefore f(x) = \frac{3}{2} \sum_{n=1}^{\infty} n \left( \frac{x}{2} \right)^{n-1} \left( \frac{1}{2} \right) = \sum_{n=1}^{\infty} \frac{3n}{2^{n+1}} x^{n-1}$$

Differentiating term by term preserves the radius of convergence. Since $f(x)$ is the derivative of a geometric series that converges for $|x/2| < 1$, we conclude that the radius of convergence is $2$.

(b) 

$$\lim_{x \to 0} \left( \frac{\cos x - 1 - x^2}{5x^2} \right) = \lim_{x \to 0} \left( \frac{1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots - 1 - x^2}{5x^2} \right)$$

$$= \lim_{x \to 0} \left( \frac{-\frac{3x^2}{2} + \frac{x^4}{24} - \cdots}{5x^2} \right) = \frac{-3}{10}$$

3. (21 points) Find the radius of convergence and the interval of convergence of the power series. Show all work and state any theorems or tests that you use.

$$\sum_{n=2}^{\infty} \frac{(5x - 2)^n}{\ln n}$$
Solution:

\[
\left| \frac{a_{n+1}}{a_n} \right| = \frac{|5x - 2|^{n+1}}{\ln(n+1)} \cdot \frac{\ln n}{|5x - 2|^n} = \frac{\ln n}{\ln(n+1)} |5x - 2| \to \frac{1/n}{1/(n+1)} |5x - 2| \quad \text{(L'H)}
\]

\[
= \left( \frac{n+1}{n} \right) |5x - 2| \to |5x - 2| \text{ as } n \to \infty.
\]

By the Ratio Test, the series converges for all \( x \) such that \( |5x - 2| < 1 \):

\[
|5x - 2| < 1,
\]

\[
\left| x - \frac{2}{5} \right| < \frac{1}{5} \implies \text{R.O.C.} = \frac{1}{5}
\]

If \( x = 1/5 \) then we have \( \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} \), which is convergent by the Alternating Series Test. If \( x = 3/5 \), then we have \( \sum_{n=2}^{\infty} \frac{1}{\ln n} > \sum_{n=2}^{\infty} \frac{1}{n} \), which diverges by the Comparison Test with the harmonic series. Therefore, the interval of convergence is \( \left[ \frac{1}{5}, \frac{3}{5} \right) \).

4. (22 points) Consider the definite integral \( I = \int_0^{1/2} \ln (1 + x^2) \, dx \).

(a) Express \( I \) as an infinite series.

(b) How many terms are needed in your series from (a) to approximate \( I \) with error less than \( \frac{1}{300} \)?

Solution:

(a)

\[
I = \int_0^{1/2} \left( \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{n} \right) \, dx
\]

\[
= \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{n(2n+1)} \bigg|_0^{1/2}
\]

\[
= \sum_{n=1}^{\infty} \frac{(-1)^n}{n(2n+1)2^{2n+1}}
\]

(b) \( |a_2| = \frac{1}{2(5) \cdot 2^5} = \frac{1}{10 \cdot 32} = \frac{1}{320} < \frac{1}{300} \), so by the Alternating Series Estimation Theorem we only need to keep one term to reach the desired accuracy.