1. (10 pts ea.) Evaluate the integrals; show all work:

(a) \( \int \frac{3x + 1}{x(x+1)^2} \, dx \)  
(b) \( \int \frac{\sin^5 x}{\cos^4 x} \, dx \)  
(c) \( \int_0^{\pi/2} \frac{\cos q}{\sqrt{1 + \sin^2 q}} \, dq \)

**Solution:** (a) Use a partial fraction decomposition

\[
\frac{3x + 1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \Rightarrow A(x^2+2x+1)+B(x^2+x)+C(x) = 3x+1 \Rightarrow A = 1, B = -1, C = 2
\]

\[
\int \frac{3x + 1}{x(x+1)^2} \, dx = \int \frac{1}{x} + \frac{-1}{x+1} + \frac{2}{(x+1)^2} \, dx = \ln|\,| - \ln|\,| + \frac{2}{x+1} + C
\]

(b) \( \int \frac{\sin^5 x}{\cos^4 x} \, dx \)  

Use a trig-sub with \( \sin(q) = \tan(\theta) \) and \( \cos(q) \, dq = \sec^2(\theta) \, d\theta \).

\[
\int \frac{\cos(q)}{\sqrt{1 + \sin^2(q)}} \, dq = \int \frac{\sec^2(\theta)}{\sqrt{1 + \tan^2(\theta)}} \, d\theta = \int \frac{\sec^2 \theta}{\sec \theta} \, d\theta = \ln |\sec \theta + \tan \theta| + C = \ln \left| \frac{\sin^2(q) + 1 + \sin(q)}{\cos(q)} \right| + C
\]

So adding the endpoints give us the final answer...

\[
= \ln \left| \frac{\sin^2(q) + 1 + \sin(q)}{\cos(q)} \right| \bigg|_{\pi/2}^{\pi} = \ln \left| \frac{\sin^2(\pi/2) + 1 + \sin(\pi/2)}{\cos(\pi/2)} \right| - \ln \left| \frac{\sin^2(0) + 1 + \sin(0)}{\cos(0)} \right| = \ln \left| \frac{\sqrt{2} + 1}{1} \right| - \ln \left| \frac{\sqrt{0} + 1}{0} \right| = \ln \left| \sqrt{2} + 1 \right| - \ln |1|
\]

2. You have just been hired by the Prime Theme Park to help them decide what new ride to build. They have a current profit model \( p(t) = 9t^{1/3} \). If they choose to build Princess Toni’s Magical Wonder Ride, the profit curve would be \( T(t) = \pi \sin(\frac{\pi t}{8}) + 9t^{1/3} \). If instead they choose to build Meredith’s Whirly Twirly Swing Ride, the profit model would be \( M(t) = 3\ln(t) + 9t^{1/3} \). The board has asked you to decide which ride they should build to maximize profit from months \( t = 1 \) to \( t = 4 \).

(a) (16 pts) Calculate the exact value of the increase in profit for each of the two proposed rides.
(b) (4 pts) Which ride should the theme park build? (You might need that \( \ln(4) \approx 1.5 \) for this part.)

**Solution:** (a) Increased profit when adding Tony’s ride is \( \int_1^4 T(t) - p(t) \, dt \).

\[
\int_1^4 \pi \sin(\frac{\pi t}{8}) + 9t^{1/3} - 9t^{1/3} \, dt = \int_1^4 \pi \sin(\frac{\pi t}{8}) \, dt = -\pi \left( \frac{8}{\pi} \right) \cos(\frac{\pi t}{8}) \bigg|_1^4 = -8 \cos(\frac{\pi}{2}) + 8 \cos(\frac{\pi}{8}) = 8 \cos(\frac{\pi}{8})
\]
Increased profit when adding Meredith’s ride is \[ \int_1^4 M(t) - p(t) dt. \] Use IBP with \( u = \ln(t) \) and \( dv = 3t \ dt \).

\[ \int_1^4 3\ln(t) + 9t^{1/3} - 9t^{1/3} dt = \int_1^4 3\ln(t) dt - \int_3 t^{1/3} dt = 3\ln(t)|_1^4 - 3t^{4/3}|_1^4 = 12\ln(4) - 12 + 3 = 12\ln(4) - 9 \]

(b) Using that \( \ln(4) \approx 1.5 \) and \( \cos(x) \leq 1 \), \( 8 \cos(\pi/8) \leq 8 \) and \( 12\ln(4) - 9 \approx 9 \) so the theme park should build Meredith’s ride.

3. (a) (6 pts) Given the profit function \( p(t) = 9t^{1/3} \), use the Midpoint Rule with \( n = 3 \) equal subintervals to approximate the profit earned from \( t = 1 \) to \( t = 4 \) months.

(b) (6 pts) Give a reasonable bound on the error in the approximation from (a). **You do not need to simplify your answer**

(c) (6 pts) Determine the minimum integer number of subintervals necessary to ensure the approximation from (a) has error \( |E_M| < \frac{3}{24} \).

**Solution:**

(a) The 3 subintervals are \([1,2], [2,3] \) and \([3,4]\), the midpoints of which are 1.5, 2.5 and 3.5, and \( \Delta x = 1 \). Thus, \[ M_3 = \left( \frac{3}{2} \right)^{1/3} + \left( \frac{5}{2} \right)^{1/3} + \left( \frac{7}{2} \right)^{1/3} \]

(b) \( |E_M| \leq \frac{K(b - a)^3}{24n^2} \), so we need to find \( K \).

\( p(t) = 9t^{1/3} \rightarrow p'(t) = 3t^{-2/3} \rightarrow p''(t) = -2t^{-5/3} \)

so \( |p''(t)| = \frac{2}{t^{5/3}} \leq 2 = K \) for \( 1 \leq t \leq 4 \).

With \( n = 3 \) and \([a, b] = [1, 4]\), we get \( |E_M| \leq \frac{(2)(4 - 1)^3}{24(3)^2} = \frac{2}{24} = \frac{1}{12} \)

(c) \( |E_M| \leq \frac{(2)(3^3)}{24n^2} < \frac{3}{24} \rightarrow \frac{2(9)}{n^2} < 1 \rightarrow 18 < n^2 \)

and the smallest integer for which this is true is \( n = 5 \)

4. (a) (10 pts) Does the integral \( \int_1^\infty \frac{1}{x^2 \arctan x} \ dx \) converge or diverge?

(b) (10 pts) For what value of \( \lambda \) does the following integral converge to 1? \( \int_0^\infty e^{-\lambda x} \ dx \) (\( \leftarrow \) 1)

**Solution:**

(a) \( 0 \leq \frac{1}{x^2 \arctan x} \leq \frac{1}{x^2 \arctan(1)} \) and \( \int_1^\infty \frac{1}{x^2} dx \) is known to be a convergent \( p \)-integral \((p > 1)\), therefore our integral is **CONVERGENT**

(b) \( 1 \leftarrow \int_0^\infty e^{-\lambda x} dx = \lim_{t \to \infty} \int_0^t e^{\lambda x} dx = \lim_{t \to \infty} \left( -\frac{1}{\lambda} e^{-\lambda x} \right) \bigg|_0^t = \lim_{t \to \infty} -\frac{1}{\lambda} (e^{-\lambda t} - 1) = -\frac{1}{\lambda} (0 - 1) \)

\( \rightarrow 1 = \frac{1}{\lambda} \rightarrow \lambda = 1 \)

**Solution:** (\leftarrow ignore this)
5. Short answer (4 pts ea.)

(a) What is the partial fractions decomposition of \( \frac{1}{(x+1)(x^5-x)} \)? Do not attempt to determine the values of the coefficients.

(b) If \( f(x) \leq g(x) \) and \( \int_0^\infty g(x) \, dx \) is divergent, then give 2 reasons why we cannot conclude anything about \( \int_0^\infty f(x) \, dx \).

(c) It is true that \( \int_{-\infty}^{\infty} f(x) \, dx = \lim_{t \to -\infty} \int_0^t f(x) \, dx + \lim_{s \to \infty} \int_s^t f(x) \, dx \) (if the right-hand side limits exist), but it is not true that \( \int_{-\infty}^{\infty} f(x) \, dx = \lim_{t \to \infty} \int_{-t}^t f(x) \, dx \). Illustrate this using a simple function.

Solution: (a) \[
\frac{1}{(x+1)(x^5-x)} = \frac{1}{(x+1)x(x^4-1)} = \frac{1}{(x+1)(x^2-1)(x^2+1)}
\]

\[
= \frac{1}{x(x-1)(x+1)(x^2+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} + \frac{D}{x^2+1} + \frac{Ex+F}{x^2-1}
\]

(b) (1) to use the Comparison Test at all, you need to know that \( 0 \leq f(x) \leq g(x) \), and (2) even if (1) was satisfied, we can’t conclude anything if the larger integral is divergent.

(c) Using \( f(x) = x \), and doing the problem correctly, we would have: \( \lim_{t \to -\infty} \int_0^t x \, dx = \lim_{t \to -\infty} \frac{1}{2} x^2 \bigg|_0^t = \lim_{t \to -\infty} -\infty = -\infty \), so \( \text{DIVERGENT} \).

But also: \( \lim_{t \to \infty} \int_{-t}^t x \, dx = \lim_{t \to \infty} \frac{1}{2} x^2 \bigg|_{-t}^t = \lim_{t \to \infty} \frac{1}{2} (t^2 - (-t)^2) = \lim_{t \to \infty} \frac{1}{2} (0) = 0, \text{ “CONVERGENT”} \).