1. (10 pts) Given the differential equation \( \frac{dS}{dr} + \frac{2}{r}S = 0 \) with initial value \( S(2) = 3 \), and \( r, S > 0 \), solve for \( S \) and simplify.

Solution:

\[
\frac{dS}{dr} + \frac{2}{r}S = 0 \\
\frac{dS}{dr} = -\frac{2}{r}S \\
\int \frac{dS}{S} = \int -\frac{2}{r} dr \\
\ln S = -2 \ln r + C \quad \text{(since } r, S > 0) \\
S = e^{-2 \ln r + C} = Ae^{-2 \ln r} = Ar^{-2}
\]

Use the initial value \( S(2) = 3 \) to find that \( 3 = A(2)^{-2} \Rightarrow A = 12 \).

\[
S = 12r^{-2}
\]

2. Parts (a) and (b) are not related.

(a) (13 pts) The curve \( y = \cosh(x) \), \( 0 \leq x \leq 2 \), is rotated about the \( y \)-axis. Set up an integral to find the area of the generated surface, then evaluate the integral.

\( \text{(Hint: Recall that } \cosh^2(x) - \sinh^2(x) = 1. \)\)

Solution:

\[
y = \cosh x, \quad y' = \sinh x \\
S = \int_a^b 2\pi r \, ds = \int_a^b 2\pi r \sqrt{1 + (y')^2} \, dx = \int_0^2 2\pi x \sqrt{1 + \cosh^2 x} \, dx \\
= \int_0^2 2\pi x \underbrace{\cosh x \, dx}_{u = \cosh x, \, du = \sinh x} = 2\pi \left( \left[ x \sinh x \right]_0^2 - \int_0^2 \sinh x \, dx \right) \\
= 2\pi \left( \left[ x \sinh x - \cosh x \right]_0^2 \right) = 2\pi \left( 2 \sinh 2 - \cosh 2 - (0 - \cosh 0) \right) \\
= 2\pi \left( 2 \sinh 2 - \cosh 2 + 1 \right)
\]
(b) (18 pts) The semicircular arc shown at right extends from point \((-4, 3)\) to point \((5, 0)\).

i. Set up (but do not evaluate) an integral to find the length of the arc.

ii. Now consider the shaded region shown at right, bounded by the arc and a line segment connecting the endpoints of the arc. Set up (but do not evaluate) an integral to find the volume of the solid generated by rotating the region about the line \(x = -5\).

Solution:

i. (9 pts) The equation of the semicircle is \(y = \sqrt{25 - x^2}\).

\[
y' = \frac{-2x}{2\sqrt{25 - x^2}} = \frac{-x}{\sqrt{25 - x^2}}
\]

\[
L = \int_a^b ds = \int_a^b \sqrt{1 + (y')^2} \, dx = \int_{-4}^5 \sqrt{1 + \frac{x^2}{25 - x^2}} \, dx
\]

ii. (9 pts) The equation of the line is \(y = -x/3 + 5/3\). By the Shell Method,

\[
V = \int_a^b 2\pi rh \, dx = \int_{-4}^5 2\pi (x + 5) \left(\sqrt{25 - x^2} - (-x/3 + 5/3)\right) \, dx
\]

3. (21 pts) Determine whether the following are convergent or divergent. If convergent, find the value the sequence or series converges to. Justify your work and name any relevant tests or theorems.

(a) \(\left\{\frac{3 + \cos^2 m}{2^m}\right\}\)  
(b) \(\left\{n^2 \arctan \left(\frac{2}{n^2}\right)\right\}\)  
(c) \(\sum_{k=1}^{\infty} \sin \left(\frac{k}{2k + 1}\right)\)

Solution:

(a) (7 pts)

\[
0 \leq \cos^2 m \leq 1 \\
3 \leq 3 + \cos^2 m \leq 4 \\
\frac{3}{2^m} \leq \frac{3 + \cos^2 m}{2^m} \leq \frac{4}{2^m}
\]

By the Squeeze Theorem, since \(\lim_{m \to \infty} \frac{3}{2^m} = \lim_{m \to \infty} \frac{4}{2^m} = 0\), the sequence \(\left\{\frac{3 + \cos^2 m}{2^m}\right\}\) converges to 0.
(b) \(\lim_{n \to \infty} n^2 \arctan(2/n^2) = \lim_{n \to \infty} \frac{\arctan(2/n^2)}{1/n^2} = \lim_{n \to \infty} \frac{1}{1+4/n^4} \cdot \frac{-4}{n^4} = \lim_{n \to \infty} \frac{2}{1+4/n^4} = \frac{2}{5} \)

(c) (7 pts) Test for Divergence:

\[ \lim_{k \to \infty} \sin \left( \frac{k}{2k+1} \right) = \sin \left( \lim_{k \to \infty} \frac{k}{2k+1} \right) = \sin \frac{1}{2} \neq 0. \]

The series is **divergent**.

4. (20 pts) Consider the squares shown below. The largest square has a side length of 1 m, the next largest square has a side length of \(\frac{2}{3}\) m, and the next has a side length of \(\frac{4}{9}\) m, etc. Each square has \(\frac{2}{3}\) the side length of the previous one. Suppose there is an infinite number of squares and \(S\) represents the total area of all the squares.

![Diagram of squares](image)

(a) Let \(a_n\) represent the area of the \(n^{th}\) square. Find an expression for \(a_n\).

(b) Is \(a_n\) bounded? If so, find bounds for \(a_n\) and justify. If not, explain why not.

(c) Does the sequence \(\{\sqrt{a_n}\}\) converge? If so, what does it converge to? If not, explain why not.

(d) The total area \(S\) corresponds to the sum of the series \(\sum_{n=1}^{\infty} a_n\). Let \(s_n\) equal the \(n^{th}\) partial sum of the series. Find \(s_3\) and leave your answer unsimplified.

(e) Find the value of \(S\) or explain why it does not exist.

**Solution:**

(a) (4 pts) \(a_n = \left(\frac{4}{9}\right)^{n-1}\)

(b) (4 pts) Yes, \(a_n\) is bounded. It is bounded above by \(1\), the area of the first square, and bounded below by \(0\) since the area of each square is positive.

(c) (4 pts) \(\lim_{n \to \infty} \sqrt{a_n} = \lim_{n \to \infty} \sqrt{\left(\frac{4}{9}\right)^{n-1}} = \lim_{n \to \infty} (2/3)^{n-1} = 0\) because this is an \(r^n\) sequence with \(r = 2/3 < 1\).

(d) (4 pts) \(s_3 = a_1 + a_2 + a_3 = 1 + \frac{4}{9} + \frac{16}{81}\)

(e) (4 pts) This is a geometric series with first term \(a = 1\) and ratio \(4/9\):

\[ S = \frac{a}{1-r} = \frac{1}{1-4/9} = \frac{9}{5}. \]
5. (18 pts) The following problems are not related.

(a) Suppose the centroid of a region is located at \((6, 4)\) and the density of the region is \(\rho = 2\). If the moment of the region about the \(y\)-axis is 300, what is the area of the region?

(b) When a particle is at \((x, 0)\), it is attracted toward the origin with a force of \(F(x) = -k/x^2\), where \(k\) is a constant. Find the work done on the particle if it moves from \(x = b\) to \(x = a\), \(0 < a < b\).

(c) Suppose \(\sum_{n=1}^{\infty} c_n = 15\). Let \(s_n\) equal the \(n\)th partial sum of the series. Does the sequence \(a_n = 3s_n + 100c_n\) converge or diverge? If it converges, find the value it converges to. If the answer cannot be determined, write “not enough info”.

**Solution:**

(a) (6 pts) 
\[
\bar{x} = \frac{M_y}{m} = \frac{M_y}{\rho A} \implies 6 = \frac{300}{2A} \implies A = \frac{150}{6} = 25.
\]

(b) (6 pts) 
\[
W = \int_{b}^{a} F(x) \, dx = \int_{b}^{a} -\frac{k}{x^2} \, dx = \left[ -\frac{k}{x} \right]_{b}^{a} = \frac{k}{a} - \frac{k}{b}.
\]

Note: A negative answer also was acceptable.

(c) (6 pts) Since \(\sum c_n\) converges to 15, \(\lim_{n \to \infty} c_n = 0\) and \(\lim_{n \to \infty} s_n = 15\). Then
\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} (3s_n + 100c_n) = 3(15) + 100(0) = 45.
\]