1. (30 points) Evaluate the following (show all of your work):

(a) \( \int_1^2 \frac{\sqrt{x^2-1}}{x} \, dx \).

(b) \( \int x^5 \ln x \, dx \).

Solution:

(a) \( \int_1^2 \frac{\sqrt{x^2-1}}{x} \, dx \) - use a trig substitution

Let \( x = \sec \theta \). Then \( dx = \sec(\theta) \tan(\theta) \, d\theta \). The integral becomes

\( \int_\frac{\pi}{3}^0 \tan^2(\theta) \, d\theta \), which equals \( \int_0^\frac{\pi}{3} \tan(\theta) \, d\theta \), which evaluates to \( \sqrt{3} - \frac{\pi}{3} \).

(b) \( \int x^5 \ln x \, dx \) - use integration by parts

With \( u = \ln x \) and \( dv = x^5 \, dx \), we get

\[ \frac{x^6}{6} \ln x - \int \frac{x^6}{6} \, dx = \frac{x^6}{6} \ln x - \frac{x^6}{36} + C \]

2. (30 points) A bowl is created by rotating the arc of the curve \( y = \frac{x^2}{3} \) between \((0, 0)\) and \((6, 12)\) around the \(y\)-axis.

(a) Set up, but do not evaluate, an integral to find the surface area of the bowl.

(b) The bowl is filled with water to depth \( h_0 \). What is the volume of water in the bowl? (Your answer will contain \( h_0 \)).

(c) Suppose that the water is evaporating from the bowl in such a way that the following equation is satisfied:

\[ \frac{dh}{dt} = -kh^2 \]

where \( k > 0 \) is a constant and \( h(t) \) is the depth of the water at time \( t \). Find \( h(t) \) given that the water starts at depth \( h_0 \). Fully simplify your answer.

Solution:
(a) Surface Area \( \int 2\pi r \, ds \), where \( r = x \) and \( ds = \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \)

Thus surface area = \( \int_0^6 2\pi x \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = \int_0^6 2\pi x \sqrt{1 + \frac{4x^2}{9}} \, dx = \frac{9\pi}{4} \int_1^{17} u^{1/2} \, du \)

= \( \frac{3\pi}{2} u^{3/2} \bigg|_1^{17} = \frac{3\pi}{2} (17^{3/2} - 1) \)

(b) Using disks: \( r = x = \sqrt{3y} \)

Thus Vol = \( \int_0^{h_0} \pi r^2 \, dy = \int_0^{h_0} \pi 3y \, dy = \pi \frac{3y^2}{2} \bigg|_0^{h_0} = \frac{3}{2} \pi (h_0)^2 \)

(c) Thus Vol = \( \int 2\pi rh \, dx = \int_0^{\sqrt{3h_0}} 2\pi x (h_0 - \frac{x^2}{3}) \, dx = \pi (x^2 h_0 - \frac{x^4}{6}) \bigg|_0^{\sqrt{3h_0}} = \frac{3}{2} \pi (h_0)^2 \)

3. (40 points) There are two unrelated parts to this question.

(a) Consider the sequence \( a_n = \ln(n) \frac{n^3}{n^4 + 2} \).

(i) Show that the sequence \( a_n \) converges to zero.
(ii) Show that the series \( \sum_{n=1}^{\infty} a_n \) diverges.

(b) Consider the function given by

\[ f(x) = \sum_{k=0}^{\infty} \frac{(-1/4)^k}{k!} (x + 2)^k \]

(i) Derive the radius of convergence of \( f(x) \).
(ii) Find \( f(-2) \).
(iii) Find \( f^{(25)}(-2) \), that is \( \frac{d^{25}f}{dx^{25}} \) at \( x = -2 \).

**Solution:**

(a) (i) Note \( \ln(n) \frac{n^3}{n^4 + 2} < \ln(n) \frac{n^3}{n^4} = \ln(n)/n \). Then L’Hospital’s rule applied to \( f(x) = \ln(x)/x \) implies \( \lim_{x \to \infty} f(x) = \lim_{x \to \infty} 1/x = 0 \), so \( a_n \to 0 \).

(ii) Note \( \ln(n) \frac{n^3}{n^4 + 2} \geq \frac{n^3}{n^4} \). Then,

\[ \frac{n^3}{n^4 + 2} = \frac{n^4}{n^4 + 2} \]

has limit 1 as \( n \to \infty \) (dominance of powers, or divide through by \( n^4 \)). By the limit comparison test and direct comparison test, the series diverges.
(b) (i) The ratio test suggests
\[ \left| \frac{(-1/4)^{k+1}(x + 2)^{k+1}}{(k + 1)!} \frac{k!}{(-1/4)^k(x + 2)^k} \right| = \frac{|x + 2|}{4k} \to 0 \]
as \( k \to \infty \). Thus the radius of convergence is \( \infty \).

(ii) \( f(-2) = (-1/4)^0/(0!) = 1 \).

(iii) Equate \((-1)^{25}/(4^{25}25!)) = f^{(25)}(-2)/(25!) \) yielding \( f^{(25)}(-2) = -(1/4)^{25} \).

4. (30 points) Consider the curve \( C \) defined by the parametric equations
\[ x = \frac{1}{2} \left( t + \frac{1}{t} \right) \quad \text{and} \quad y = \frac{1}{2} \left( t - \frac{1}{t} \right) \]
with \( 0.1 \leq t \leq 10 \).

(a) Compute \( \frac{dy}{dx} \) in terms of the parameter \( t \).

(b) Find the points (if any) on \( C \) where the tangent is vertical or horizontal.

(c) Is \( C \) a portion of a conic section? If yes, which one? (Hint: Compute \((x - y)(x + y)\))

(d) Set up, but do not evaluate, an integral to find the arc length of \( C \).

Solution:

(a) We evaluate
\[ \frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{1}{2} \left( 1 + t^{-2} \right) \]
\[ \frac{1}{2} (1 - t^{-2}) \]

(b) In part (a), the numerator is never zero and the denominator is zero only if \( t = \pm 1 \), of which only \( t = 1 \) belongs to the range of \( t \) used to define the curve \( C \). Thus, there are no horizontal tangents and there is a vertical tangent at \( t = 1 \).

(c) Since
\[ x - y = t^{-1} \quad \text{and} \quad x + y = t \]
we have
\[ x^2 - y^2 = 1 \]
and the answer is yes, \( C \) is part of a hyperbola, a conic section.

(d) The arc length \( L \) of \( C \) is
\[ L = \int_{0.1}^{10} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt = \int_{0.1}^{10} \sqrt{\left( \frac{1}{2} (1 - t^{-2}) \right)^2 + \left( \frac{1}{2} (1 + t^{-2}) \right)^2} \, dt. \]

5. (20 points) The following problems are unrelated.

(a) Sketch the polar curve \( r = \sin^2(\theta) \) for \( 0 \leq \theta \leq 2\pi \). Include axis labels and an arrow denoting the direction of the curve.
(b) The pie dish shown below is 9 inches across the top, 7 inches across the bottom and 3 inches deep. The goal of this problem is to approximate the volume of the pie dish using \( n \) horizontal slices. Let the \( y \) axis be vertical, the \( x \) axis be horizontal, and place the origin at the center of the bottom of the dish. Thus, the equation for the line defining the right edge of the dish is \( y = 3(x - 7/2) \). Determine if the following Riemann sum, using a partition of \( n \) subintervals of size \( \Delta y \), is the correct approximation, and if not, fix it so that it is correct:

\[
\text{Volume} \approx n \sum_{i=1}^{n} \pi \left(3 \left(y_i - \frac{7}{2}\right)\right)^2 \Delta y,
\]

where \( y_i \) is a sample point on the \( i \)th subinterval of the partition.

**Solution:**

(a) The polar curve is

\[
\sum_{i=1}^{n} \pi \left(\frac{1}{3} y_i + \frac{7}{2}\right)^2 \Delta y
\]
Have a wonderful break!

**Some Trigonometric identities**

\[ 2 \cos^2(x) = 1 + \cos(2x) \]
\[ 2 \sin^2(x) = 1 - \cos(2x) \]
\[ \sin(2x) = 2 \sin(x) \cos(x) \]
\[ \cos(2x) = \cos^2(x) - \sin^2(x) \]

**Inverse Trigonometric Integral Identities**

\[
\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}(u/a) + C, \quad u^2 < a^2 \\
\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1}(u/a) + C \\
\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1}|u/a| + C, \quad u^2 > a^2
\]

**Center of Mass Integrals**

\[
M = \rho \int_{a}^{b} (f(x) - g(x)) \, dx \\
M_x = \frac{\rho}{2} \int_{a}^{b} (f(x) - g(x))^2 \, dx \\
M_y = \rho \int_{a}^{b} x(f(x) - g(x)) \, dx \\
\bar{x} = \frac{M_y}{M} \quad \text{and} \quad \bar{y} = \frac{M_x}{M}
\]

**Frequently Used Maclaurin Series**

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad R = 1 \\
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad R = \infty \\
\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad R = \infty \\
\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n!} \quad R = \infty \\
\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad R = 1 \\
\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \quad R = 1 \\
(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \quad R = 1
\]

**Midpoint Rule**

\[
\int_{a}^{b} f(x) \, dx \approx \Delta x \left[ f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n) \right] \quad \text{where} \quad \Delta x = \frac{b-a}{n} \quad \text{and} \quad \bar{x}_i = \frac{x_i-1 + x_i}{2} \quad \text{and} \quad |E_M| \leq \frac{K(b-a)^3}{24n^2}.
\]

**Trapezoidal Rule**

\[
\int_{a}^{b} f(x) \, dx \approx \frac{\Delta x}{2} \left[ f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n) \right] \quad \text{where} \quad \Delta x = \frac{b-a}{n} \quad \text{and} \quad |E_T| \leq \frac{K(b-a)^3}{12n^2}.
\]