1. (30 points) Evaluate the following (show all of your work):

   (a) \( \int_1^2 \frac{\sqrt{x^2 - 1}}{x} \, dx \).

   (b) \( \int x^5 \ln x \, dx \).

2. (30 points) A bowl is created by rotating the arc of the curve \( y = \frac{x^2}{3} \) between \((0, 0)\) and \((6, 12)\) around the \(y\)-axis.

   (a) Set up, but do not evaluate, an integral to find the surface area of the bowl.

   (b) The bowl is filled with water to depth \( h_0 \). What is the volume of water in the bowl? (Your answer will contain \( h_0 \)).

   (c) Suppose that the water is evaporating from the bowl in such a way that the following equation is satisfied:

   \[ \frac{dh}{dt} = -kh^2 \]

   where \( k > 0 \) is a constant and \( h(t) \) is the depth of the water at time \( t \). Find \( h(t) \) given that the water starts at depth \( h_0 \). Fully simplify your answer.

3. (40 points) There are two unrelated parts to this question.

   (a) Consider the sequence

   \[ a_n = \ln(n) \frac{n^3}{n^4 + 2} \]

   (i) Show that the sequence \( a_n \) converges to zero.

   (ii) Show that the series \( \sum_{n=1}^{\infty} a_n \) diverges.
(b) Consider the function given by

\[ f(x) = \sum_{k=0}^{\infty} \frac{(-1/4)^k}{k!} (x + 2)^k \]

(i) Derive the radius of convergence of \( f(x) \).

(ii) Find \( f(-2) \).

(iii) Find \( f^{(25)}(-2) \), that is \( \frac{d^{25}f}{dx^{25}} \) at \( x = -2 \).

4. (30 points) Consider the curve \( C \) defined by the parametric equations

\[ x = \frac{1}{2} \left( t + \frac{1}{t} \right) \quad \text{and} \quad y = \frac{1}{2} \left( t - \frac{1}{t} \right) \]

with \( 0.1 \leq t \leq 10 \).

(a) Compute \( \frac{dy}{dx} \) in terms of the parameter \( t \).

(b) Find the points (if any) on \( C \) where the tangent is vertical or horizontal.

(c) Is \( C \) a portion of a conic section? If yes, which one? (Hint: Compute \( (x - y)(x + y) \))

(d) Set up, but do not evaluate, an integral to find the arc length of \( C \).

5. (20 points) The following problems are unrelated.

(a) Sketch the polar curve \( r = \sin^2(\theta) \) for \( 0 \leq \theta \leq 2\pi \). Include axis labels and an arrow denoting the direction of the curve.

(b) The pie dish shown below is 9 inches across the top, 7 inches across the bottom and 3 inches deep. The goal of this problem is to approximate the volume of the pie dish using \( n \) horizontal slices. Let the \( y \) axis be vertical, the \( x \) axis be horizontal, and place the origin at the center of the bottom of the dish. Thus, the equation for the line defining the right edge of the dish is \( y = 3(x - 7/2) \). Determine if the following Riemann sum, using a partition of \( n \) subintervals of size \( \Delta y \), is the correct approximation, and if not, fix it so that it is correct:

\[ \text{Volume} \approx \sum_{i=1}^{n} \pi (3(y_i - 7/2))^2 \Delta y, \]

where \( y_i \) is a sample point on the \( i \)th subinterval of the partition.
Some Trigonometric identities

\[ 2 \cos^2(x) = 1 + \cos(2x) \]
\[ 2 \sin^2(x) = 1 - \cos(2x) \]
\[ \sin(2x) = 2 \sin(x) \cos(x) \]
\[ \cos(2x) = \cos^2(x) - \sin^2(x) \]

Inverse Trigonometric Integral Identities

\[ \int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}(u/a) + C, \quad u^2 < a^2 \]
\[ \int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1}(u/a) + C \]
\[ \int \frac{du}{u \sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1}|u/a| + C, \quad u^2 > a^2 \]

Center of Mass Integrals

\[ M = \rho \int_a^b (f(x) - g(x)) \, dx \]
\[ M_x = \frac{\rho}{2} \int_a^b (f^2(x) - g^2(x)) \, dx \]
\[ M_y = \rho \int_a^b x(f(x) - g(x)) \, dx \]
\[ \bar{x} = \frac{M_y}{M} \quad \text{and} \quad \bar{y} = \frac{M_x}{M} \]

Midpoint Rule

\[ \int_a^b f(x) \, dx \approx \Delta x \left[ f(x_1) + f(x_2) + \cdots + f(x_n) \right] \] where \( \Delta x = \frac{b - a}{n} \) and \( \bar{x}_i = \frac{x_{i-1} + x_i}{2} \) and \( |E_M| \leq \frac{K(b - a)^3}{24n^2} \).

Trapezoidal Rule

\[ \int_a^b f(x) \, dx \approx \frac{\Delta x}{2} \left[ f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n) \right] \] where \( \Delta x = \frac{b - a}{n} \) and \( |E_T| \leq \frac{K(b - a)^3}{12n^2} \).

Frequently Used Maclaurin Series

\[ \frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n \quad R = 1 \]
\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad R = \infty \]
\[ \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad R = \infty \]
\[ \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad R = \infty \]
\[ \tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad R = 1 \]
\[ \ln(1 + x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \quad R = 1 \]
\[ (1 + x)^k = \sum_{n=0}^{\infty} \left( \begin{array}{l} k \\ n \end{array} \right) x^n \quad R = 1 \]