Exam 3
Wednesday, November 29, 2017

This exam is worth 100 points and has 4 questions.

- On the front of your bluebook, write your name, a grading key, your instructor name, and section number.
- Show all work! Answers with no justification will receive no points.
- Begin each problem on a new page.
- No notes or papers, calculators, cell phones, or electronic devices are permitted.

Frequently Used Maclaurin Series

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<th>Maclaurin Series</th>
<th>Radius of Convergence</th>
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<td>[ \frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n ]</td>
<td>[ R = 1 ]</td>
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<tr>
<td>[ (1 + x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n ]</td>
<td>[ R = 1 ]</td>
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<tr>
<td>[ \tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n + 1} ]</td>
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<tr>
<td>[ \ln(1 + x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} ]</td>
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<td>[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} ]</td>
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<td>[ \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n + 1)!} ]</td>
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<tr>
<td>[ \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} ]</td>
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</table>
1. (20 points) Determine whether the following series are absolutely convergent, conditionally convergent, or divergent. Justify your answer and name any tests used.

(a) \( \sum_{n=1}^{\infty} \frac{\cos (\pi n)}{n^2} \)

(b) \( \sum_{n=100}^{\infty} \frac{n}{2n+3} \)

Solution:

(a) \( \sum_{n=1}^{\infty} \left| \frac{\cos (\pi n)}{n^2} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \), which is a convergent p-series \((p = 3.14 > 1)\). Thus, the series is absolutely convergent by the Direct Comparison Test.

(b) \( \sum_{n=100}^{\infty} \frac{n}{2n+3} \) diverges by the Test for Divergence because \( \lim_{n \to \infty} \frac{n}{2n+3} = \frac{1}{2} \neq 0 \).

2. (30 points) This problem has two unrelated parts.

(a) Find the interval of convergence of \( \sum_{n=135}^{\infty} \frac{(x - 2)^n}{n^2} \).

(b) Suppose you want to find a series representation of the function \( \frac{1}{3-x} \):

i. Professor Kleiber rewrites this function as \( \frac{1}{1 - (x - 2)} \) and uses the geometric series (the first series in the table provided), to find a series representation of the function. Write his series using sigma notation and find the interval of convergence.

ii. Professor Lyles rewrites this function as \( \frac{1/3}{1 - (x/3)} \) and then uses the geometric series to find a series representation of the function. Write her series using sigma notation and find the interval of convergence.

iii. Why don’t these two representations and intervals agree? Is one wrong?

Solution:

(a) Use the ratio test,

\[ \lim_{n \to \infty} \left| \frac{(x - 2)^{n+1} \frac{n^2}{(n+1)^2 (x - 2)^n}}{(x - 2)^n \frac{n^2}{(n+1)^2}} \right| = \lim_{n \to \infty} \left| \frac{(x - 2) \frac{n^2}{(n+1)^2}}{n^2} \right| = |x - 2| \]

which converges for all \( x \) such that \(-1 < x - 2 < 1\), i.e., \( 1 < x < 3 \). We need to check the endpoints: \( x = 3 \) implies \( \sum_{n=135}^{\infty} \frac{1}{n^2} \) which converges by the p-test with \( p = 2 \); \( x = 1 \) implies \( \sum_{n=135}^{\infty} \frac{(-1)^n}{n^2} \) which converges absolutely by the p-test with \( p = 2 \). Thus, the interval of convergence is \( 1 \leq x \leq 3 \).
(b) i. Kleiber’s solution:
\[
\frac{1}{1 - (x - 2)} = \sum_{n=0}^{\infty} (x - 2)^n
\]
and because this is a geometric series, has interval of convergence \( |x - 2| < 1 \), i.e., \( 1 < x < 3 \).

ii. Lyles’ solution:
\[
\frac{1/3}{1 - (x/3)} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty} \frac{1}{3^{n+1}} x^n
\]
and because this is a geometric series, has interval of convergence \( |x/3| < 1 \), i.e., \( -3 < x < 3 \).

iii. These are two different power series expansions of \((3 - x)^{-1}\); the first is centered at 2, the second at 0. Neither is wrong, they are both valid representations.

3. (30 points) This problem has two unrelated parts.

(a) Give the Maclaurin series for \( f(x) = x^2e^{-x^2} \), and state the radius of convergence of the series. Give your final answer in sigma notation, fully simplified.

(b) Consider the function given by
\[
f(x) = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{k!}{(2k)!} (x - 3)^k
\]

i. Find \( f(3) \).

ii. Find \( f^{(100)}(3) \), that is, \( \frac{d^{100}f}{dx^{100}} \) at \( x = 3 \).

iii. Find the Taylor series for \( f(3x) \) about \( x = 1 \). Give your answer in sigma notation.

Solution:

(a) \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, R = \infty \)
thus
\( e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}, = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}, R = \infty \)
thus
\[ x^2e^{-x^2} = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n!}, R = \infty \]

(b) i. \( f(3) = -1 \)

ii. Since \( f(x) \) equals the given power series, we know this series is the Taylor series for \( f(x) \) centered at \( x = 3 \).
In other words,
\[ f(x) = \sum_{k=0}^{\infty} (-1)^{(k+1)} \frac{k!}{(2k)!} (x - 3)^k \]
\[
= \sum_{k=0}^{\infty} c_k (x - 3)^k, \text{ where } c_k = \frac{f^{(k)}(3)}{k!}.
\]
So to find the 100th derivative, we equate coefficients of the term \((x - 3)^{100}\), i.e.

\[
(-1)^{101} \frac{100!}{200!} = c_{100} = \frac{f^{(100)}(3)}{100!}
\]

Thus

\[
f^{(100)}(3) = -\frac{(100!)^2}{200!}
\]

iii. \(f(3x) = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{k!}{(2k)!} \cdot (3x - 3)^k\)

\[
= \sum_{k=0}^{\infty} (-1)^{k+1} \frac{k!3^k}{(2k)!} (x - 1)^k,
\]

which is the Taylor series for \(f(3x)\) centered at \(x = 1\).

4. (20 pts) Consider the function given by \(f(x) = 6 \sin x + 1 - x^2\)

(a) Find \(T_2(x)\), the (degree \(n = 2\)) Taylor polynomial for \(f(x)\) centered at \(a = 0\).

(b) Use Taylor’s Formula to find an upper bound for the absolute value of the error in using \(T_2(x)\) from part (a) to approximate \(f\left(\frac{1}{10}\right)\).

**Solution:**

(a) Since

\[
6 \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!} \cdot x^{2n+1}
\]

we have

\[
T_2(x) = 1 + 6x - x^2
\]

(b) We use Taylor’s formula to write

\[
\left| R_3 \left( \frac{1}{10} \right) \right| = \left| \frac{f^{(3)}(z)}{6} \left( \frac{1}{10} \right)^3 \right| = \left| f^{(3)}(z) \right| \frac{1}{6} \frac{1}{10^3},
\]

where \(0 < z < \frac{1}{10}\). Since

\[
f^{(3)}(x) = -6 \cos x,
\]

and \(\cos x\) is decreasing on \((0, \frac{1}{10})\), we obtain

\[
\left| f^{(3)}(z) \right| \leq \left| f^{(3)}(0) \right| \leq 6,
\]

and substituting in Taylor’s formula

\[
\left| R_3 \left( \frac{1}{10} \right) \right| \leq 10^{-3}.
\]

Thus, an upper bound is

\[
10^{-3}
\]