1. (20 points) Evaluate the following:
   (a) \( \int \frac{3}{(x - 1)(x^2 + 2)} \, dx \)

(b) Solve for \( y \) in the differential equation: \( \frac{dy}{dx} = xy \sin x \) with initial value \( y(\pi) = 1 \).

Solution:

(a) Use partial fractions.

\[
\int \frac{3}{(x - 1)(x^2 + 2)} \, dx = \int \left( \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 2} \right) \, dx
\]

\[
A(x^2 + 2) + (Bx + C)(x - 1) = 3
\]

\[
(A + B)x^2 + (-B + C)x + (2A - C) = 3
\]

\[
A + B = 0, \quad -B + C = 0, \quad 2A - C = 3
\]

\[
A = 1, \quad B = -1, \quad C = -1
\]

\[
\int \left( \frac{1}{x - 1} - \frac{x + 1}{x^2 + 2} \right) \, dx = \int \left( \frac{1}{x - 1} - \frac{x}{x^2 + 2} - \frac{1}{u^{2}+2} \right) \, dx
\]

\[
= \ln |x - 1| - \frac{1}{2} \ln |x^2 + 2| - \frac{1}{\sqrt{2}} \arctan \left( \frac{x}{\sqrt{2}} \right) + C
\]

(b) This is a separable equation.

\[\frac{dy}{dx} = xy \sin x \Rightarrow \int \frac{dy}{y} = \int \frac{x \sin x \, dx}{u} \]
Use integration by parts with \( u = x, \ du = dx, \) and \( dv = \sin x \, dx, \ v = -\cos x. \)

\[
\ln |y| = -x \cos x + \int \cos x \, dx = -x \cos x + \sin x + C
\]
\[
|y| = e^{-x \cos x + \sin x + C}
\]

Plug in the initial value \((\pi, 1)\).

\[
1 = e^{\pi + C} \Rightarrow C = -\pi
\]
\[
y = e^{-x \cos x + \sin x - \pi}
\]

2. (14 points) Consider the region shown at right. Set up, but do not evaluate, the integrals to calculate:

(a) The volume generated by rotating this region about the line \(x = 3.\)

(b) Consider the upper curve \(y = x^2/2,\) defined on the interval shown on the graph. Find the surface area of the solid generated by rotating the curve about the \(y\)-axis. (The lower curve is not relevant for this part.)

Solution:

(a) The curves intersect at \((0, 0)\) and \((2, 2)\). By the Shell Method,

\[
V = \int 2\pi rh \, dx = \int_0^2 2\pi (3 - x) \left( \frac{x^2}{2} - \frac{x^4}{4} - \frac{x^2}{2} \right) \, dx
\]

(b)

\[
S = \int 2\pi r \sqrt{1 + (dy/dx)^2} \, dx = \int_0^2 2\pi x \sqrt{1 + x^2} \, dx
\]

3. (24 points) Do the following converge or diverge? Explain your work and name any test or theorem you use.

(a) \(\int_0^\infty \frac{dx}{1 + e^x}\)  
(b) \(\left\{ \frac{(\ln m)^2}{m} \right\}_{m=15}^\infty\)  
(c) \(\sum_{n=3}^\infty \frac{\sqrt{n+2}}{n-2}\)

Solution:

(a) Since \(0 < \frac{1}{1 + e^x} < \frac{1}{e^x}\) and \(\int_0^\infty \frac{dx}{e^x}\) is convergent, by the Comparison Theorem \(\int_0^\infty \frac{dx}{1 + e^x}\) also is convergent.
Alternate solution:
\[
\int_0^\infty \frac{dx}{1 + e^x} = \lim_{t \to \infty} \int_0^t \frac{dx}{1 + e^x}
\]

Let \( u = 1 + e^x \), \( du = e^x \, dx \).

\[
= \lim_{t \to \infty} \int_2^{1+e^t} \frac{1+e^t}{u(u-1)} \, du = \lim_{t \to \infty} \left[ \ln |u-1| - \ln |u| \right]_2^{1+e^t}
\]

\[
= \lim_{t \to \infty} \left[ \ln \left( \frac{u-1}{u} \right) \right]_2^{1+e^t} = \lim_{t \to \infty} \left( \ln \frac{e^t}{1+e^t} - \ln \frac{1}{2} \right) = \ln 1 + \ln 2 = \ln 2
\]

The integral is \( \text{convergent} \).

(b) \( \lim_{m \to \infty} a_m = \lim_{m \to \infty} \frac{(\ln m)^2}{m} L' H = \lim_{m \to \infty} \frac{2 \ln m}{m} = \lim_{m \to \infty} \frac{2}{m} = 0 \)

The sequence \( \text{converges} \) to 0.

(c) Since \( \sqrt{n+2} > \sqrt{n} = \frac{1}{\sqrt{n}} \) and \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \) is a divergent p-series, by the Comparison Test \( \sum_{n=3}^{\infty} \frac{\sqrt{n+2}}{n-2} \) also is \( \text{divergent} \).

Alternate solution: Use the Limit Comparison Test with \( b_n = 1/\sqrt{n} \).

\[
\lim_{n \to \infty} \left| \frac{a_n}{b_n} \right| = \lim_{n \to \infty} \left| \frac{\sqrt{n+2}}{n-2} \cdot \frac{\sqrt{n}}{1} \right| = \lim_{n \to \infty} \sqrt{\frac{n^2 + 2n}{n^2 - 2n + 4}} = 1
\]

Since \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \) is a divergent p-series, \( \sum_{n=3}^{\infty} \frac{\sqrt{n+2}}{n-2} \) also is \( \text{divergent} \).

4. (16 points) Find the values of \( x \) for which the following series converge absolutely, converge conditionally, or diverge: (a) \( \sum_{n=2}^{\infty} \frac{(x+3)^n}{n^2 - 1} \) (b) \( \sum_{n=0}^{\infty} \frac{3^n}{n!} x^{2n} \)

Solution:

(a) Use the Ratio Test.

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(x+3)^{n+1}}{(n+1)^2 - 1} \cdot \frac{n^2 - 1}{(x+3)^n} \right| = |x + 3| < 1
\]

The series is absolutely convergent on \(-4 < x < -2\). Now check the endpoints. When \( x = -2, \) the series \( \sum_{n=2}^{\infty} \frac{1}{n^2 - 1} \) converges by the Limit Comparison Test with the convergent
p-series. When $x = -4$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} - 1$ converges by the Alternating Series Test since $\frac{1}{(n+1)^2 - 1} < \frac{1}{n^2 - 1}$ and $\lim_{n \to \infty} \frac{1}{n^2 - 1} = 0$.

Therefore the series $\sum_{n=2}^{\infty} \frac{(x+3)^n}{n^2 - 1}$ is absolutely convergent for $-4 \leq x \leq -2$ is not conditionally convergent for any $x$, and is divergent for $x < -4, x > -2$.

(b) Use the Ratio Test.

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{3^{n+1}x^{2n+2}}{(n+1)!} \cdot \frac{n!}{3^n x^{2n}} = \lim_{n \to \infty} \frac{3x^2}{n+1} = 0$$

Since the ratio is 0, the series is absolutely convergent for all $x$ and not conditionally convergent or divergent for any $x$.

5. (14 points) Let $a_1 = 1, a_2 = \frac{1 + 1/4}{1 + 1/2}, a_3 = \frac{1 + 1/4 + 1/4^2}{1 + 1/2 + 1/2^2}, a_4 = \frac{1 + 1/4 + 1/4^2 + 1/4^3}{1 + 1/2 + 1/2^2 + 1/2^3}, \ldots$

(a) Does the sequence $\{a_n\}$ converge? If yes, find its limit. Explain.

(b) Does $\sum_{n=1}^{\infty} a_n$ converge? Explain your work and state any test or theorem you use.

Solution:

(a) $a_n$ is the quotient of geometric series. The numerator has ratio $1/4$ and the denominator has ratio $1/2$. The sum of an infinite geometric series is $S = a/(1 - r)$.

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1 + 1/4 + \frac{1}{4^2} + \cdots + \frac{1}{4^{n-1}}}{1 + 1/2 + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}} = \lim_{n \to \infty} \frac{1 + 1/4 + \cdots + 1/4^{n-1}}{1 + 1/2 + \cdots + 1/2^{n-1}} = \frac{1/(1 - 1/4)}{1/(1 - 1/2)} = \frac{2}{3}$$

(b) By the Test for Divergence, since $\lim_{n \to \infty} a_n \neq 0$, the series diverges.

6. (20 points)

(a) Find a Maclaurin series for $x \sin(x^2)$.

(b) Use your answer from part (a) to find a series for $\int x \sin(x^2) \, dx$.

(c) Use your answer from part (b) to find the 8th order Taylor polynomial $T_8(x)$ for the integral.

(d) Use $T_8(x)$ to approximate the value of $\int_0^1 x \sin(x^2) \, dx$.

(e) Is your approximation an overestimate or underestimate? Explain your reasoning.
Solution:

(a) \[ \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} \]

(b) \[ \int x \sin(x^2) \, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+4}}{(2n+1)!(4n+4)} \]

(c) \[ T_8(x) = C + \frac{x^4}{4} - \frac{x^8}{48} \]

(d) \[ \int_0^1 x \sin(x^2) \, dx \approx T_8(1) - T_8(0) = \frac{1}{4} - \frac{1}{48} = \frac{11}{48} \]

(e) Because the series is alternating, the approximation is an underestimate.

7. (14 points)

(a) Sketch the graphs of \( r = -2 \cos \theta \) and \( r = 1 \).

(b) Find the area of the region inside \( r = -2 \cos \theta \) and outside \( r = 1 \).

Solution:

(a)

(b) We wish to find the area of the shaded region. The curve \( r = -2 \cos \theta \) (for \( \theta = 0 \) to \( \pi \)) and curve \( r = 1 \) (for \( \theta = 0 \) to \( 2\pi \)) intersect at \( -2 \cos \theta = 1 \Rightarrow \theta = 2\pi/3 \). We double the area of the shaded region above the \( x \)-axis.

\[
A = 2 \int_\frac{\pi}{2}^{\frac{\pi}{2}} \left( r_1^2 - r_2^2 \right) \, d\theta = 2 \int_\frac{2\pi}{3}^{\frac{\pi}{2}} \frac{1}{2} \left( (-2 \cos \theta)^2 - 1^2 \right) \, d\theta = 2 \int_\frac{2\pi}{3}^{\frac{\pi}{2}} (4 \cos^2 \theta - 1) \, d\theta \\
= \int_\frac{2\pi}{3}^{\frac{\pi}{2}} (2(1 + \cos 2\theta) - 1) \, d\theta = \int_\frac{2\pi}{3}^{\frac{\pi}{2}} (1 + 2 \cos 2\theta) \, d\theta \\
= \left[ \theta + \sin 2\theta \right]^{\frac{\pi}{2}}_{\frac{2\pi}{3}} = \left( \frac{\pi}{2} - \left( \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right) \right) = \frac{\pi}{3} + \frac{\sqrt{3}}{2} \]
8. (16 points) Write the word TRUE (if always true) or FALSE. For this problem, no explanation is necessary.

(a) If \( \sum_{n=1}^{\infty} a_n \) diverges, then \( \sum_{n=1}^{\infty} |a_n| \) diverges.

(b) The length of the curve \( r^2 = 2 \cos \theta \) is given by \( L = \int_{0}^{\pi/2} 4 \sqrt{2 \cos \theta + \frac{\sin^2 \theta}{2 \cos \theta}} \, d\theta \).

(c) If the parametric equations \( x = f(t) \) and \( y = g(t) \) are twice differentiable, then \( \frac{d^2 y}{dx^2} = \frac{d^2 y/\,dt^2}{d^2 x/\,dt^2} \).

(d) The second order Taylor polynomial \( T_2(x) \) centered at \( a = 1 \) is used to approximate \( f(x) = \sqrt{x} \) on the interval \([1, 1.1]\). An estimate for the error of the approximation using Taylor's Formula is \( \frac{1}{16} (0.1)^3 \).

Solution:

(a) True because if \( \sum_{n=1}^{\infty} |a_n| \) converges, then so does \( \sum_{n=1}^{\infty} a_n \).

(b) True. Note that \( r \) is not defined for \( \cos \theta < 0 \). If \( r = \sqrt{2 \cos \theta} \), then \( \frac{dr}{\,d\theta} = \frac{-\sin \theta}{\sqrt{2 \cos \theta}} \). The length of the entire curve is four times the length for \( \theta = 0 \) to \( \pi/2 \).

\[
L = \int_{0}^{\pi/2} \sqrt{r^2 + \left( \frac{dr}{\,d\theta} \right)^2} \, d\theta = 4 \int_{0}^{\pi/2} \sqrt{2 \cos \theta + \frac{\sin^2 \theta}{2 \cos \theta}} \, d\theta.
\]

(c) False. \( \frac{d^2 y}{dx^2} = \frac{d}{\,dt} \left( \frac{dy}{dx} \right) \frac{dx}{\,dt} \)

(d) True. \( R_2(x) = \frac{f'''(z)}{3!} (x - 1)^3 \) for \( z \) between \( x \) and 1. On the interval \([1, 1.1]\), \( |f'''(z)| = \left| \frac{3}{8x^3} \right| \leq \frac{3}{8} \) and \( |(x - 1)^3| \leq (0.1)^3 \), so \( |R_2(x)| \leq \frac{3}{8} \cdot \frac{1}{6} \cdot (0.1)^3 = \frac{1}{16} (0.1)^3 \).
9. (12 points) Match the graphs shown below to the following equations. No explanation is necessary.

(a) \(x^2 = 1 + y^2/2\)
(b) \(x^2 = 1 - y^2/4\)
(c) \(x = 2 \cos t, y = \sin t\)
(d) \(r^2 = 4/\theta\)
(e) \(r^2 = (\cos \theta)/4\)
(f) \(r = 1/2 + \cos(2\theta)\)

Solution: (1) d (2) f (3) c (4) a