On the front of your bluebook, please write: a grading key, your name, student ID, and section and instructor. This exam is worth 100 points and has 7 questions.

- **Show all work!** Answers with no justification will receive no points.
- Please begin each problem on a new page.
- No notes, calculators, or electronic devices are permitted.

1. (12 points) Determine whether each of the following series is absolutely convergent, conditionally convergent, or divergent. Explain your work and name any test or theorem you use.

(a) \[ \sum_{n=1}^{\infty} \frac{5 \cdot 10 \cdot 15 \cdots (5n)}{n!} \]

(b) \[ \sum_{k=5}^{\infty} \frac{(-1)^k}{\ln(3k)} \]

**Solution:**

(a) There are two ways to verify that this series diverges. The first is to observe

\[ \sum_{n=1}^{\infty} \frac{5 \cdot 10 \cdot 15 \cdots (5n)}{n!} = \sum_{n=1}^{\infty} \frac{5^n n!}{n!} = \sum_{n=1}^{\infty} 5^n \]

so the series diverges by the Test for Divergence.

Alternatively, you can also use the ratio test:

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{5 \cdot 10 \cdots (5n)(5(n+1))}{(n+1)!} \cdot \frac{n!}{5 \cdot 10 \cdots (5n)} = \lim_{n \to \infty} \frac{5(n+1)}{n+1} = 5
\]

Since the limit of the ratio is greater than 1, the series diverges. (Similar to HW 8.4 #39)

(b) Use the Alternating Series Test. Observe that \( a_n = \frac{1}{\ln(3n)} \) is positive for \( k \geq 5 \), it’s decreasing since \( a_n = \frac{1}{\ln(3n)} > \frac{1}{\ln(3(n+1))} = a_{n+1} \), and \( \lim_{n \to \infty} a_n = 0 \). Since the hypotheses of AST are satisfied, we conclude that \( \sum_{k=5}^{\infty} \frac{(-1)^k}{\ln(3k)} \) converges.

We note that the series does not converge absolutely. There are several ways to justify this statement. One is to note that \( \frac{1}{k} < \frac{1}{\ln(3k)} \) and \( \sum_{k=5}^{\infty} \frac{1}{k} \) diverges so \( \sum_{k=5}^{\infty} \frac{1}{\ln(3k)} \) diverges by the Comparison Test.

Therefore, \( \sum_{k=5}^{\infty} \frac{(-1)^k}{\ln(3k)} \) converges conditionally.
2. (24 points) For each of the following series, answer the following questions: (i) For what values of \( x \) does the series converge conditionally? converge absolutely? (ii) For what values of \( x \) does the series diverge? (iii) Determine the interval of convergence and the radius of convergence. Explain your work and name any test or theorem you use.

(a) \( \sum_{n=0}^{\infty} \frac{(2x - 5)^n}{\sqrt{n} + 3} \)

(b) \( \sum_{n=1}^{\infty} (-1)^n(n + 2)!(8x - 1)^{n+1} \)

Solution:

(a) Start with the ratio test:

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(2x - 5)^{n+1}}{\sqrt{n + 1} + 3} \cdot \frac{\sqrt{n} + 3}{(2x - 5)^n} = \lim_{n \to \infty} \frac{\sqrt{n} + 3}{\sqrt{n} + 1 + 3} |2x - 5| = |2x - 5|
\]

|2x - 5| < 1 implies that 2 < \( x \) < 3.

At endpoint \( x = 3 \), the series becomes \( \sum_{n=0}^{\infty} \frac{1}{\sqrt{n} + 3} \). We note that \( \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}} \) diverges since it is a p-series with \( p = 1/2 \). And, since \( \lim_{n \to \infty} \frac{\sqrt{n} + 3}{\sqrt{n}} = 1 \), the original series diverges by the limit comparison test.

At endpoint \( x = 2 \), the series becomes \( \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n} + 3} \) which converges by the Alternating Series Test. (Note: To verify the hypotheses of AST, we check \( a_n = \frac{1}{\sqrt{n} + 3} \) is positive, decreasing, and \( \lim_{n \to \infty} a_n = 0 \).) Further, we note that the series does not converge absolutely since the absolute value of the terms is the same series as when \( x = 3 \). Therefore, the series converges conditionally at \( x = 2 \).

Answers: The series converges absolutely for 2 < \( x \) < 3, it converges conditionally for \( x = 2 \) and it diverges for all other values of \( x \) (i.e. it diverges for \( x < 2 \) and \( x \geq 3 \)). The interval of convergence is 2 ≤ \( x \) < 3 and the radius of convergence is \( R = 1/2 \).

(b) The ratio test gives:

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(n + 3)!(8x - 1)^{n+2}}{(n + 2)!(8x - 1)^{n+1}} = \lim_{n \to \infty} \frac{|(n + 3)(8x - 1)|}{(n + 2)!} = \infty
\]

for all values of \( x \) except \( x = 1/8 \). Therefore, the series diverges for all \( x \) except \( x = 1/8 \). At \( x = 1/8 \) the series converges absolutely. There are no \( x \) values where the series converges conditionally. The interval of convergence is just the point \( x = 1/8 \) and the radius of convergence is 0.
3. (12 points) Start with the geometric series \( \sum_{n=0}^{\infty} \left( \frac{x}{3} \right)^{n} \), \(-3 < x < 3\).

(a) Find the sum of the series as a function of \( x \).

(b) Find the sum of the series \( \sum_{n=1}^{\infty} n \left( \frac{x}{3} \right)^{n} \).

(c) Use your answer from part (b) to find the sum of \( \sum_{n=1}^{\infty} \frac{n}{6^n} \).

Solution:

(a) For \(-3 < x < 3\), the geometric series \( \sum_{n=0}^{\infty} \left( \frac{x}{3} \right)^{n} = \frac{1}{1 - \frac{x}{3}} = \frac{3}{3 - x} \)

(b) Term-by-term differentiation yields:
\[
\frac{d}{dx} \left( \sum_{n=1}^{\infty} n \left( \frac{x}{3} \right)^{n-1} \right) = \sum_{n=1}^{\infty} n \left( \frac{x}{3} \right)^{n-1} \left( \frac{1}{3} \right) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{3^n} = \frac{d}{dx} \left( \frac{3}{3 - x} \right) = \frac{3}{(3 - x)^2}
\]

Multiplying both sides of \( \sum_{n=1}^{\infty} \frac{nx^{n-1}}{3^n} = \frac{3}{(3 - x)^2} \) by \( x \) yields
\[
\sum_{n=1}^{\infty} \frac{nx^n}{3^n} = \frac{3x}{(3 - x)^2}
\]

(c) Substitute \( x = 1/2 \) into your answer from part (b) to obtain:
\[
\sum_{n=1}^{\infty} \frac{n}{6^n} = \frac{3(1/2)}{(3 - (1/2))^2} = \frac{6}{25}
\]

(Similar to HW 8.6 #40)

4. (12 points) Start with the Maclaurin series for \( e^x \) and the definition of \( \cosh x = \frac{e^x + e^{-x}}{2} \) to answer the following questions.

(a) Find the Maclaurin series for \( \cosh(2x) \).

(b) Now, use your answer from part (a) to find the first three nonzero terms of the Maclaurin series for \( \cosh^2(2x) \).

Solution:
(a)

\[ e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \]

\[ \cosh x = \frac{e^x + e^{-x}}{2} \]

\[ = \frac{1}{2} \left[ \sum_{k=0}^{\infty} \frac{x^k}{k!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!} \right] \]

\[ = \frac{1}{2} \left[ (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots) + (1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots) \right] \]

\[ = \frac{1}{2} \left[ 2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \cdots \right] \]

\[ = \sum_{k=0}^{\infty} \frac{(2x)^{2k}}{(2k)!} \]

Then,

\[ \cosh(2x) = \sum_{k=0}^{\infty} \frac{(2x)^{2k}}{(2k)!} \]

(b)

\[ \cosh^2(2x) = \left( 1 + \frac{2x^2}{2!} + \frac{2^4x^4}{4!} + \frac{2^6x^6}{6!} + \cdots \right) \left( 1 + \frac{2x^2}{2!} + \frac{2^4x^4}{4!} + \frac{2^6x^6}{6!} + \cdots \right) \]

\[ = 1 + 2^2x^2 + \left( \frac{2^4}{4!} + \frac{2^4}{2!2!} + \frac{2^4}{4!} \right)x^4 + \cdots \]

\[ = 1 + 4x^2 + \frac{16}{3}x^4 + \cdots \]

Therefore, the first three nonzero terms of the Maclaurin series for \( \cosh^2(2x) \) is \( 1 + 4x^2 + \frac{16}{3}x^4 \).

5. (10 points)

(a) Give the definition of the Taylor series, centered at \( x = a \) for a function \( y = f(x) \). Assume \( y = f(x) \) has derivatives of all orders in an interval centered at \( x = a \).

(b) If \( y = f(x) \) has derivatives of all orders and if it has a Taylor series \( \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2n+1} \), what is \( f^{(4)}(0) \)?

Solution:

(a) The Taylor series of \( f \) centered at \( a \) is

\[ \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = f(a) + \frac{f'(a)}{1!} (x-a) + \frac{f''(a)}{2!} (x-a)^2 + \frac{f'''(a)}{3!} (x-a)^3 + \cdots . \]
(b) The given Taylor series is centered at \( a = 0 \) and equals \(-\frac{1}{3}x^2 + \frac{1}{5}x^4 - \ldots\). The fourth degree term \( \frac{1}{5}x^4 \) must match the Taylor series term \( \frac{f^{(4)}(0)}{4!}x^4 \).

Therefore \( f^{(4)}(0) = \frac{4!}{5} = \frac{24}{5} \).

6. (14 points) Consider the integral \( \int_{0}^{1} \frac{1}{(1 + x^2)^{1/3}} \, dx \).

(a) Use the first three nonzero terms of the Maclaurin series for \( \frac{1}{(1 + x^2)^{1/3}} \) to estimate the value of the integral.

(b) Estimate the error in your approximation in part (a).

Solution:

(a)

\[
\frac{1}{(1 + x^2)^{1/3}} = (1 + x^2)^{-1/3} = \sum_{n=0}^{\infty} \left(\frac{-1/3}{n}\right) (x^2)^n \\
= \left(\frac{-1/3}{0!}\right) + \left(\frac{-1/3}{1!}\right)x^2 + \left(\frac{-1/3}{2!}\right)x^4 + \left(\frac{-1/3}{3!}\right)x^6 + \ldots \\
= 1 + \frac{-1/3}{1!}x^2 + \frac{-1}{3^2 2!}x^4 + \frac{-1}{3^3 3!}x^6 + \ldots \\
= 1 - \frac{1}{3}x^2 + \frac{1}{3^2 2!}x^4 - \frac{1}{3^3 3!}x^6 + \ldots \\
= 1 - \frac{1}{3}x^2 + \frac{2}{9}x^4 - \frac{14}{81}x^6 + \ldots
\]

\[
\int_{0}^{1} \frac{1}{(1 + x^2)^{1/3}} \, dx = \left[ x - \frac{1}{9}x^3 + \frac{2}{45}x^5 - \frac{2}{81}x^7 + \ldots \right]_{0}^{1} \\
= 1 - \frac{1}{9} + \frac{2}{45} - \frac{2}{81} + \ldots \approx 1 - \frac{1}{9} + \frac{2}{45} = \frac{14}{15}
\]

(b) By the Alternating Series Estimation Theorem, the error using the first three terms is at most \( \frac{2}{81} \).

7. (16 points) Match the graphs shown below to the following parametric equations. Copy each graph into your bluebook. Label it with the appropriate parametric equations. Then, draw arrows on the graph to clearly show the direction in which the graph is traversed. No other explanation is required.

(a) \( x = \sqrt{t}, \ y = \sqrt{16 - t}, \ 0 \leq t \leq 16 \)

(b) \( x = t^2, \ y = t^2 - t + 1, \ -2\pi \leq t \leq 2\pi \)

(c) \( x = 4 \sin t, \ y = \cos t, \ 0 \leq t \leq 2\pi \)

(d) \( x = t \sin^2 t, \ y = t \cos^2 t, \ -2\pi \leq t \leq 2\pi \)
Solution:
(a) 3  (b) 2  (c) 1  (d) 4