APPM 1350 **Final Exam** **Summer 2017**

On the front of your bluebook, please write: a grading key, your name, student ID, your lecture number, and instructor. This exam is worth 100 points and has 4 questions on both sides of this paper.

- Submit this exam sheet with your bluebook. However, nothing on this exam sheet will be graded. Make sure all of your work is in your bluebook.
- **Show all work and simplify your answers!** Answers with no justification will receive no points.
- Please begin each problem on a new page.
- No notes or papers, calculators, cell phones, or electronic devices are permitted.

---

1. (24 points, 8 points each) Evaluate $\dfrac{dy}{dx}$ for the following expressions

a.) $y = \int_0^{e^x} \sec^5(v)dv$

b.) $y = 3\sin^{-1}(1+2x)$

c.) $y^x = \cosh(x)$

Solution:

a.) Let $u = e^x$, then $du/dx = e^x$, then by the Fundamental Theorem of Calculus,

$$\begin{aligned}\frac{d}{dx}\int_0^{e^x} \sec^5(x)dx &= \frac{d}{du}\int_0^{u} \sec^5(x)dx\frac{du}{dx} \\ &= \sec^5(u)e^x \\ &= \sec^5(e^x)e^x.\end{aligned}$$

b.) Using the chain rule,

$$\frac{d}{dx}\left(3\sin^{-1}(1+2x)\right) = 3\frac{1}{\sqrt{1-(1+2x)^2}}\cdot 2 = \frac{6}{\sqrt{1-(1+2x)^2}}$$

c.) We can use logarithmic and implicit differentiation to solve this problem:

$$\begin{aligned} y^x &= \cosh(x) \\ \ln(y^x) &= \ln(\cosh(x)) \\ x\ln(y) &= \ln(\cosh(x)) \\ \ln(y) + \frac{xy'}{y} &= \frac{\sinh(x)}{\cosh(x)} \\ \frac{xy'}{y} &= \tanh(x) - \ln(y) \\ y' &= \frac{y}{x}\left(\tanh(x) - \ln(y)\right) \end{aligned}$$

so

$$\frac{dy}{dx} = \frac{y}{x}\left(\tanh(x) - \ln(y)\right)$$

2.2. (16 points, 8 points each) Evaluate the following integrals

a.) $\displaystyle\int \frac{\cos(6x)}{\sin(6x)} dx$

b.) $\displaystyle\int_0^1 x^3\sqrt{1+x^4}dx$

Solution:

a.) Let $u = \sin(6x)$, then $du = 6\cos(6x)dx$ and

$$\begin{aligned}\int \frac{\cos(6x)}{\sin(6x)} dx &= \int \frac{1}{u}\frac{du}{6} \\ &= \frac{1}{6}\ln|u| + c \\ &= \frac{1}{6}\ln|\sin(6x)| + c\end{aligned}$$

b.) Let $u = 1 + x^4$, then $du = 4x^3dx$, so $x^3dx = \dfrac{du}{4}$. Note that $u(0) = 1$, $u(1) = 2$. So,

$$\begin{aligned}\int_0^1 x^3\sqrt{1+x^4}dx &= \int_1^2 u^{1/2}\frac{du}{4} \\ &= \frac{1}{4}\cdot\frac{2}{3}\, u^{3/2}\Big|_{u=1}^{2} \\ &= \frac{1}{6}(2^{3/2} - 1^{3/2}) \\ &= \frac{2\sqrt{2}-1}{6}\end{aligned}$$

3. (24 points) The following are unrelated:

a.) (8 points) Sketch the function function $f(x) = \tan(x) + \sqrt{3}$ on the interval $[-\pi, \pi]$. State and label all zeros, horizontal and vertical asymptotes, and intervals of increase/decrease on your graph.

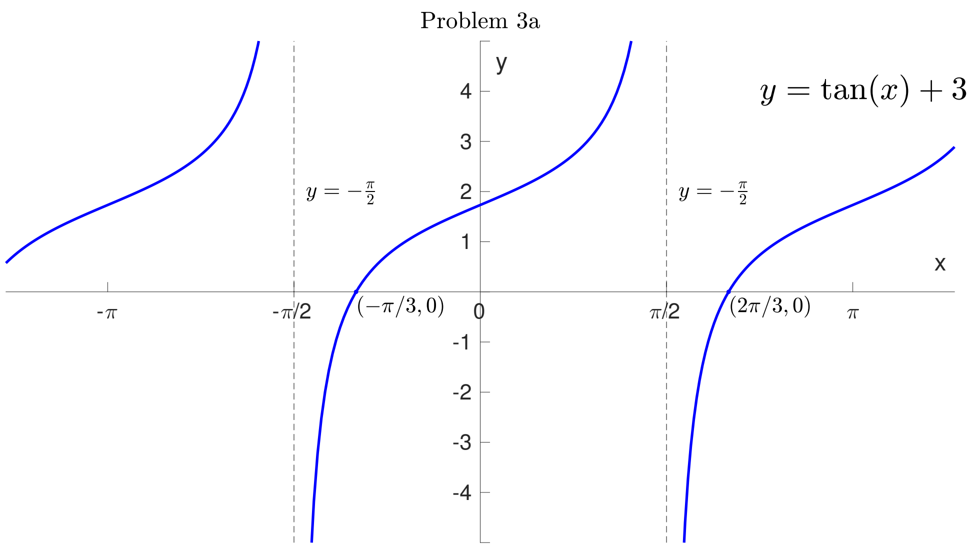
b.) (6 points) The number of bacteria in a population is given by $y(t)$ and grows by the differential equation ${}^{dy}/{}_{dt} = ky$. There are initially 50 bacteria present. After 6 hours, there are 1500 bacteria. Find the value for k.

c.) (6 points) State the domain of $y = \ln(\tan^{-1}(x))$ and find $\dfrac{dy}{dx}$

d.) (4 points) **True** or **False**: an even function is always invertible on its domain. No justification is necessary.

Solution:

a.) The graph for $f(x)$ is given below



The zeroes of the function occur when

$$\begin{aligned} \tan(x) + \sqrt{3} &= 0 \\ \tan(x) &= -\sqrt{3} \\ x &= -{}^{\pi}/{}_{3}, {}^{2\pi}/{}_{3} \end{aligned}$$

The function has vertical asymptotes at $x = -\pi/2$ and $x = \pi/2$. There are no horizontal asymptotes. As

$$\frac{df}{dx} = \sec^2(x) > 0,$$

the intervals of increase are $[-\pi, -\pi/2) \cup (-\pi/2, \pi/2) \cup (\pi/2, \pi]$, and there are no intervals of decrease.

b.) The solution to this differential equation is of the form $y = Ce^{kt}$. If there are initially 50 bacteria, then $y = 50e^{kt}$. From the given information,

$$\begin{aligned} y(6) &= 1500 = 50e^{6k} \\ \Rightarrow 30 &= e^{6k} \\ 6k &= \ln(30) \\ k &= \frac{\ln(30)}{6}\frac{1}{\text{hour}}. \end{aligned}$$

c.) $\tan^{-1}(x)$ is defined for all x, so $\ln(\tan^{-1}(x))$ is defined so long as $\tan^{-1}(x) > 0$, which occurs for $x > 0$. Thus, the domain of y is $(0, \infty)$. Using the chain rule,

$$\frac{dy}{dx} = \frac{1}{\tan^{-1}\ (x)}\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{\tan^{-1}(x)}\frac{1}{1+x^2}.$$

d.) This is **false**. An even function will not pass the horizontal line test.

4. (15 points) If 300 cm^2 of material is available to make a box with a square base and an open top, find the length of the base of the box that maximizes the volume of the box. Be sure to include all units in your final answer.

Solution:

Let h denote the height of the box and b be the length of the box's base. The practical domain for these two measurements is $h > 0$ and $b > 0$. We have a surface area for the box given by

$$S.A. = b^2 + 4hb.$$

And from the given information,

$$b^2 + 4hb = 300,$$

which suggests that

$$h = \frac{300 - b^2}{4b}$$

We want to maximize the volume of the box, given by

$$V = hb^2$$

and we can use the above equation for h to solve for V as a function of b as

$$V(b) = \frac{1}{4}(300b - b^3)$$

which has derivatives

$$\frac{dV}{db} = \frac{1}{4}(300 - 3b^2)$$
$$\frac{d^2V}{db^2} = -\frac{3}{2}b.$$

V thus has critical points when

$$\frac{1}{4}(300 - 3b^2) = 0$$
$$\Rightarrow b = \pm 10.$$

$b = -10$ is not in our practical domain, so we only consider $b = 10$cm. Note that

$$\frac{d^2V}{db^2}(10) = -\frac{3}{2}(10) < 0,$$

so our volume does have a local maximum at $b = 10$cm.

5. (21 points, 7 points each) The following are unrelated:

a.) $\lim_{x \to 2^-} \tan^{-1}\left(\dfrac{4}{(x-2)^3}\right)$

b.) $\lim_{x\to 0} \dfrac{x3^x}{9^x - 1}$

c.) Use geometry to evaluate $\int_2^9 (|2x - 10| - 6)dx$.

Solution:

a.) Let $t = \dfrac{4}{(x-2)^3}$. Note that $\lim_{x\to 2^-} \dfrac{4}{(x-2)^3} = \dfrac{4}{(-0)^3} = -\infty$, hence

$$\lim_{x \to 2^-} \tan^{-1}\left(\frac{4}{(x-2)^3}\right) = \lim_{t\to -\infty} \tan^{-1}(t) = -\pi/2.$$

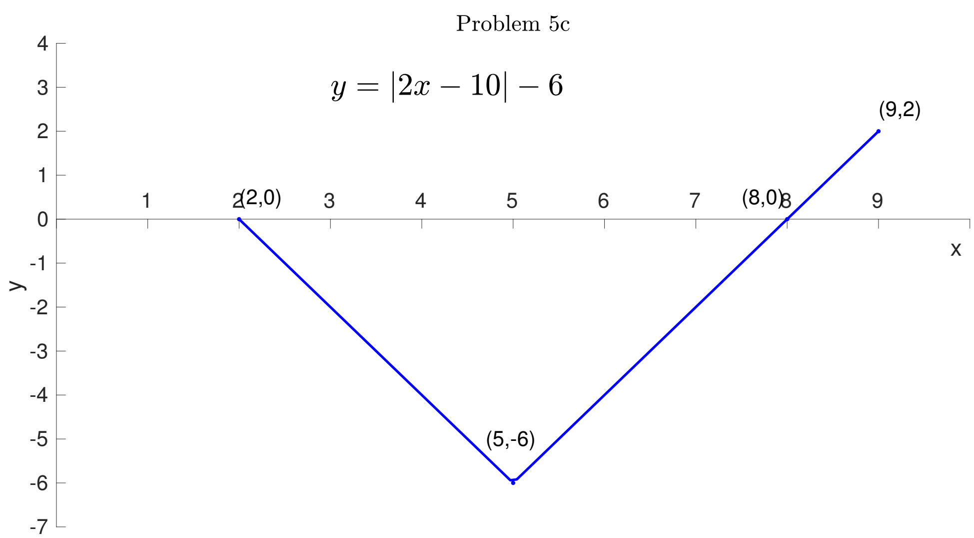
b.) Using direct evaluation,

$$\lim_{x\to 0} \frac{x3^x}{9^x - 1} = \frac{0}{0},$$

which is indeterminate. We can use L'Hospital's Rule:

$$\lim_{x\to 0} \frac{x3^x}{9^x - 1} = \lim_{x\to 0} \frac{3^x + x\ln(3)3^x}{\ln(9)9^x} = \frac{1+0}{\ln(9)} = \frac{1}{\ln(9)}$$

c.) The graph of $|2x - 10| - 6$ is given below



We thus see that the area under this curve is given by a triangle of base 6 and height 6 (and negative area) and another triangle of base 1 and height 2, so

$$\int_2^9 |2x - 10| dx = -\frac{1}{2}(6)(6) + \frac{1}{2}(1)(2) = -18 + 1 = -17$$