APPM 1350 **Exam 3** **Summer 2017**

On the front of your bluebook, please write: a grading key, your name, student ID, your lecture number, and instructor. This exam is worth 100 points and has 5 questions on both sides of this paper.

- Submit this exam sheet with your bluebook. However, nothing on this exam sheet will be graded. Make sure all of your work is in your bluebook.
- **Show all work and simplify your answers!** Answers with no justification will receive no points.
- Please begin each problem on a new page.
- No notes or papers, calculators, cell phones, or electronic devices are permitted.

The following formulas may be of use:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

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1. (28 points, 7 points each) Evaluate the following integrals using any method we've discussed in class.

a.) $\int_{-1/2}^{1/2} \sqrt{\frac{1}{4} - x^2}dx$

b.) $\int x(\sqrt{x} + \sqrt[3]{x})dx$

c.) $\int_0^{\pi/4} \sec(x)\tan(x)dx$

d.) If $\int_1^5 f(x)dx = 2$, $\int_4^5 3f(x)dx = 9$, evaluate $\int_1^4 f(x)dx$.

Solution:

a. $\sqrt{\frac{1}{4} - x^2}$ is the formula for the upper half of a circle with radius $r = 1/2$. We thus see that that given integral denotes the area of half of a circle with radius 1/2. So

$$\int_{-1/2}^{1/2} \sqrt{\frac{1}{4} - x^2}dx = \frac{1}{2}\left(\pi\left(\frac{1}{2}\right)^2\right) = \frac{\pi}{8}.$$

b.

$$\begin{aligned}\int x(\sqrt{x}+\sqrt[3]{x})dx &= \int x^{3/2}+x^{4/3}dx \\ &= \frac{x^{5/2}}{5/2}+\frac{x^{7/3}}{7/3}+c \\ &= \frac{2}{5}x^{5/2}+\frac{3}{7}x^{7/3}+c\end{aligned}$$

c.

$$\begin{aligned}\int_0^{\pi/4} \sec(x)\tan(x)dx &= \sec(x)|_{x=0}^{\pi/4} \\ &= \sec(\pi/4)-\sec(0) \\ &= \sqrt{2}-1\end{aligned}$$

d. $\int_4^5 3f(x)dx = 9$ gives that $\int_4^5 f(x)dx = 3$. So

$$\begin{aligned}\int_1^5 f(x)dx &= \int_1^4 f(x)dx + \int_4^5 f(x)dx \\ 2 &= \int_1^4 f(x)dx + 3 \\ \int_1^4 f(x)dx &= -1\end{aligned}$$

2. (15 points) If 10,800 cm^2 of material is available to make a box with a square base and an open top, find the length of the base of the box that maximizes the volume of the box. Be sure to include all units in your final answer.

Solution:

Let h denote the height of the box and b be the length of the box's base. The practical domain for these two measurements is $h > 0$ and $b > 0$. We have a surface area for the box given by

$$S.A. = b^2 + 4hb.$$

And from the given information,

$$b^2 + 4hb = 10,800,$$

which suggests that

$$h = \frac{10800 - b^2}{4b}$$

We want to maximize the volume of the box, given by

$$V = hb^2$$

and we can use the above equation for h to solve for V as a function of b as

$$V(b) = \frac{1}{4}(10800b - b^3)$$

which has derivatives

$$\frac{dV}{db} = \frac{1}{4}(10800 - 3b^2)$$
$$\frac{d^2V}{db^2} = -\frac{3}{2}b.$$

V thus has critical points when

$$\frac{1}{4}(10800 - 3b^2) = 0$$
$$\Rightarrow b = \pm 60.$$

$b = -60$ is not in our practical domain, so we only consider $b = 60$cm. Note that

$$\frac{d^2V}{db^2}(60) = -\frac{3}{2}(60) < 0,$$

so our volume does have a local maximum at $b = 60$cm.

3. (28 points) Consider the function $f(x) = \begin{cases} -|x + 1/2| & x < 0 \\ 1 - x^2 & x \geq 0 \end{cases}$

a.) (5 points) Sketch $f(x)$ on the interval $[-2, 1]$.

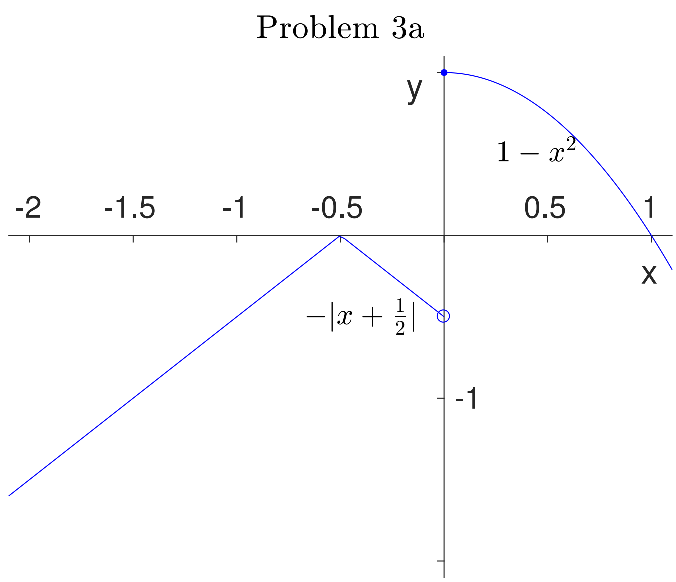
b.) (5 points) Calculate $\int_{-2}^{0} f(x)dx$ using any method we've discussed in class.

c.) (10 points) Give the expression for the right endpoint sum for a given integer n, R_n, for $f(x)$ on the interval $[0, 1]$.

d.) (5 points) Use part (c) to calculate $\int_0^1 f(x)dx$

e.) (3 points) Is $f(x)$ integrable on $[-1, 1]$? Why or why not? If so, calculate $\int_{-1}^{1} f(x)dx$

Solution:



a.)

b.) $f(x)$ on $[-2, 0]$ is made of two right triangles $f(-2) = -1.5$, so the first triangle has a height of $3/2$ and a width of $3/2$ so an area of $1/2(3/2)^2 = 9/8$. The second triangle has an area of $1/2(1/2)^2 = 1/8$. Both triangles are below the $x-$axis, so

$$\int_{-2}^{0} f(x)dx = -9/8 - 1/8 = -10/8 = -5/4.$$

c.) $\Delta x = \dfrac{1-0}{n} = \dfrac{1}{n}$ and $x_i = 0 + \dfrac{i}{n} = \dfrac{i}{n}$ so

$$\begin{aligned}
R_n &= \sum_{i=i}^{n} f(x_i)\Delta x \\
&= \sum_{i=i}^{n} \left(1 - \left(\frac{i}{n}\right)^2\right)\frac{1}{n} \\
&= \frac{1}{n}\sum_{i=1}^{n}\left(1 - \frac{i^2}{n^2}\right) \\
&= \frac{1}{n}\sum_{i=1}^{n} 1 - \frac{1}{n^3}\sum_{j=1}^{n} j^2 \\
&= 1 - \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} \\
&= 1 - \frac{2n^2+3n+1}{6n^2}
\end{aligned}$$

d.)

$$\begin{aligned}
\int_0^1 f(x)dx &= \lim_{n\to\infty} R_n \\
&= \lim_{n\to\infty}\left(1 - \frac{2n^2+3n+1}{6n^2}\right) \\
&= \lim_{n\to\infty} 1 - \lim_{n\to\infty}\frac{2 + 3/n + 1/n}{6} \\
&= 1 - 1/3 = 2/3
\end{aligned}$$

e.) Yes, $f(x)$ is integrable on $[-2, 1]$ because it has only one jump discontinuity.

$$\int_{-2}^{1} f(x)dx = \int_{-2}^{0} f(x)dx + \int_0^1 f(x)dx = -5/4 + 2/3 = -7/12.$$

4. (29 points) The following are unrelated:

a.) (12 points) Let $f(x) = \sin(3x)$. Suppose Newton's Method is used to find the root(s) of $f(x)$.
i.) Plot $f(x)$ on $[0, {}^{4\pi}/_{3}]$ and label all x and y intercepts. Sketch the tangent line at $x = {}^{\pi}/_{6}$. If we use an initial approximation of $x_1 = {}^{\pi}/_{6}$, what would our second approximation, x_2, be?
ii.) Use Newton's Method to find an expression for x_{n+1} in terms of x_n. If $x_1 = {}^{\pi}/_{4}$, find x_2.

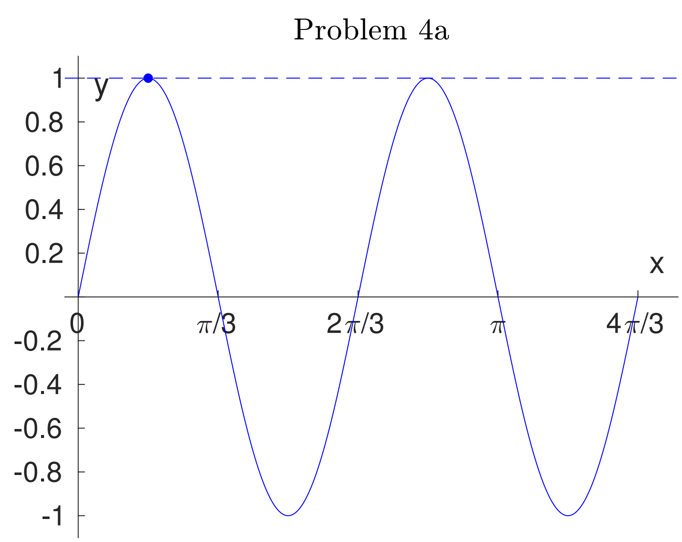
b.) (12 points) A man standing on a 32 foot cliff throws a rock up in the air with vertical velocity $v(t) = -32t + 16$ feet/second. Find $s(t)$ and calculate the displacement and total distance traveled by the rock before it hits the ground. Be sure to include all units.

c.) (5 points) Write the following expression in Sigma notation:

$$4 + 7 + 10 + 13 + 16 + 19 + 22$$

Solution

a.) i.) $f(x)$ and its tangent line are plotted below



Here, we see that $f(x)$ has a horizontal tangent line at $x = \pi/6$, so Newton's method would not provide a second approximation. Newton's Method would thus fail.

ii.) Note that $f'(x) = 3\cos(3x)$. Newton's method is given by

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{\sin(3x)}{3\cos(3x)} \\ &= x_n - \frac{1}{3}\tan(3x). \end{aligned}$$

If $x_n = \pi/4$, then

$$\begin{aligned} x_{n+1} &= \frac{\pi}{4} - \frac{1}{3}\tan({}^{3\pi}\!/\!{}_4) \\ &= \frac{\pi}{4} + \frac{1}{3} \end{aligned}$$

b.) $s(t)$ is given by the antiderivative of $v(t)$, so

$$s(t) = -16t^2 + 16t + c.$$

From the given information, $s(0) = 32$, so

$$\begin{aligned} s(t) &= -16t^2 + 16t + 32 \\ &= -16(t^2 - t - 2) \\ &= -16(t-2)(t+1) \end{aligned}$$

and we see that $s(t) = 0$ when $t = 2$. The displacement between 0 and 2 seconds is given by the difference in location, $s(t)$, at the two time point. From the given information,

$$\text{displacement} = s(2) - s(0) = 0 - 32 = -32\text{feet}.$$

Total distance traveled is defined by

$$\text{distance} = \int_0^2 |v(t)| dt$$

and note that $v(t) > 0$ on $[0, 1/2)$ and $v(t) < 0$ on $(1/2, 2]$. Therefore,

$$\begin{aligned} \text{distance} &= \int_0^{1/2} -32t + 16dt + \int_{1/2}^2 32t - 16dt \\ &= (-16t^2 + 16t)\Big|_{t=0}^{t=1/2} + (16t^2 - 16t)\Big|_{t=1/2}^{t=2} \\ &= 4 + 36 = 40\text{feet}. \end{aligned}$$

c.)

$$4 + 7 + 10 + 13 + 16 + 19 + 22 = \sum_{i=1}^{7} (1 + 3i)$$
