1. (21 points, 7 points each) Find $\dfrac{dy}{dx}$ for the following:

a.) $y = x\sec(\sqrt{x})$

b.) $4\cos(x)\sin(y^2) = 1$

c.) $y = \sqrt{\dfrac{x^2+1}{x^2+4}}$

Solution:

a.) Using the product and chain rules,

$$\begin{aligned}\frac{dy}{dx} &= x'\sec(\sqrt{x}) + x(\sec(\sqrt{x}))' \\ &= \sec(\sqrt{x}) + x\sec(\sqrt{x})\tan(\sqrt{x})(\sqrt{x})' \\ &= \sec(\sqrt{x}) + x\sec(\sqrt{x})\tan(\sqrt{x})\frac{1}{2\sqrt{x}} \\ &= \sec(\sqrt{x}) + \frac{1}{2}\sqrt{x}\sec(\sqrt{x})\tan(\sqrt{x})\end{aligned}$$

b.) By using the product rule and implicit differentiation,

$$\begin{aligned}4\cos(x)'\sin(y^2) + 4\cos(x)\sin(y^2)' &= 0 \\ -4\sin(x)\sin(y^2) + 4\cos(x)\cos(y^2)2y\frac{dy}{dx} &= 0 \\ 4\cos(x)\cos(y^2)2y\frac{dy}{dx} &= 4\sin(x)\sin(y^2) \\ \frac{dy}{dx} &= \frac{\sin(x)\sin(y^2)}{2y\cos(x)\cos(y^2)} \\ \frac{dy}{dx} &= \frac{\tan(x)\tan(y^2)}{2y}\end{aligned}$$

c.) Using the quotient and chain rule:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{2\sqrt{\dfrac{x^2+1}{x^2+4}}}\left(\frac{x^2+1}{x^2+4}\right)' \\
&= \frac{\sqrt{x^2+4}}{2\sqrt{x^2+1}}\left(\frac{(x^2+4)(2x)-(x^2+1)(2x)}{(x^2+4)^2}\right) \\
&= \frac{\sqrt{x^2+4}}{2\sqrt{x^2+1}}\left(\frac{2x^3+8x-2x^3-2x}{(x^2+4)^2}\right) \\
&= \frac{\sqrt{x^2+4}}{2\sqrt{x^2+1}}\left(\frac{6x}{(x^2+4)^2}\right) \\
&= \frac{3x}{\sqrt{x^2+1}\sqrt{(x^2+4)^3}}
\end{aligned}$$

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2. (15 points) A particle is moving along the line $y = 3x$. When $x = 4$cm, the x-coordinate of the particle's position is increasing at a rate of 2cm/min. At this moment, what is the rate of change of the particle's distance from the origin? Be sure to include all units in your answer.

Solution:

Let r denote the distance of the particle from the origin. If x and y denote the x and y position of the particle over time, then we have the equation

$$r^2 = x^2 + y^2.$$

At the time point given, we know $x = 4$cm, which means that $y = 12$cm and $r^2 = 4^2 + 12^2 = 16 + 144 = 160$ so $r = \sqrt{160} = 4\sqrt{10}$cm. We are also given that $\frac{dx}{dt} = 2$cm/min.

We know that the particle is moving along the line $y = 3x$, so we can simplify this as

$$r^2 = x^2 + (3x)^2 = 10x^2.$$

Now by using implicit differentiation, we see

$$2r\frac{dr}{dt} = 20x\frac{dx}{dt}.$$

We can now substitute in our known values to find our unknown values and find:

$$2(4\sqrt{10}\text{cm})\frac{dr}{dt} = 20(4\text{cm})\left(2\frac{\text{cm}}{\text{min}}\right)$$

so we can solve

$$\frac{dr}{dt} = \frac{20}{\sqrt{10}}\frac{\text{cm}}{\text{min}} = 2\sqrt{10}\frac{\text{cm}}{\text{min}}$$

3. (30 points) Given $f(x) = \dfrac{-x^2+x-1}{x-1}$, $f'(x) = \dfrac{x(2-x)}{(x-1)^2}$, $f''(x) = \dfrac{-2}{(x-1)^3}$,

a.) (1 point) Find the y-intercept of $f(x)$. Note that $f(x)$ does not have any x-intercepts.

b.) (5 points) Find any vertical and horizontal asymptotes of $f(x)$.

c.) (8 points) State the intervals of increase and decrease of $f(x)$ using interval notation. Find the x and y values of any local extrema.

d.) (6 points) On what intervals (using interval notation) is $f(x)$ concave up? Concave down? State the x and y values of any inflection points.

e.) (10 points) Sketch $f(x)$

Solution:

a.) the y-intercept occurs at $f(0) = \dfrac{-1}{-1} = 1$.

b.) We can look for horizontal asymptotes as

$$\lim_{x\to\pm\infty} \frac{-x^2+x-1}{x-1} = \lim_{x\to\pm\infty} \frac{-x+1-{}^1\!/x}{1-{}^1\!/x} = \mp\infty,$$

so there are no horizontal asymptotes. Looking at our denominator, we suspect a vertical asymptote at $x = 1$:

$$\lim_{x\to 1^-} \frac{-x^2+x-1}{x-1} = \frac{-1+1-1}{-0} = \frac{-1}{-0} = +\infty.$$

So we have a vertical asymptote at $x = 1$. To check what happens on the other side of 1,

$$\lim_{x\to 1^+} \frac{-x^2+x-1}{x-1} = \frac{-1+1-1}{+0} = \frac{-1}{+0} = -\infty.$$

c.) For the given $f'(x)$, we have critical points at $x = 0, 1, 2$

Intervals	y'	y
$(-\infty, 0)$	$\frac{-+}{+} = -$	decreasing
$(0, 1)$	$\frac{++}{+} = +$	increasing
$(1, 2)$	$\frac{++}{+} = +$	increasing
$(2, \infty)$	$\frac{+-}{+} = -$	decreasing

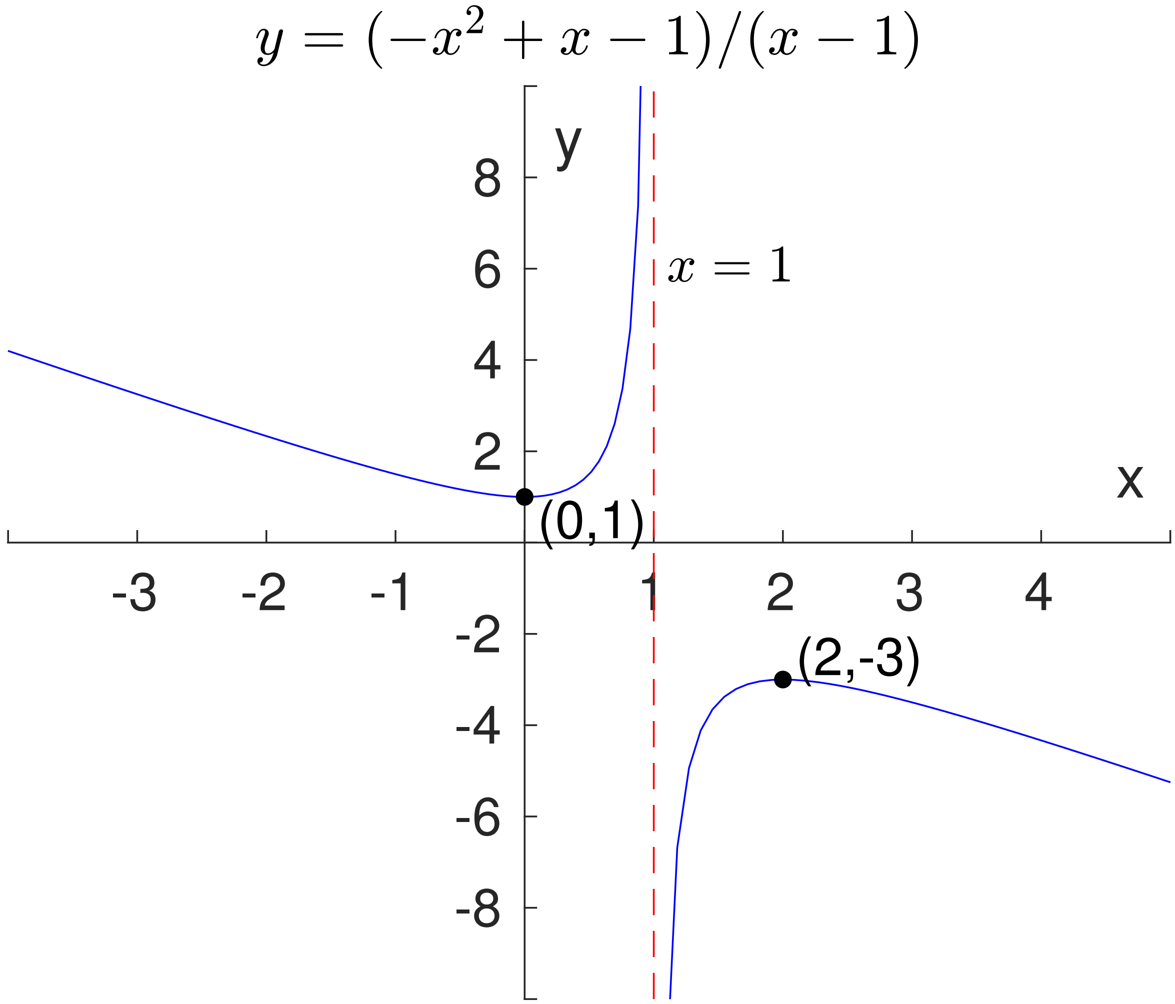
So $f(x)$ is increasing on $(0, 1) \cup (1, 2)$ and decreasing on $(-\infty, 0) \cup (2, \infty)$. We thus have a local minimum at $(0, 1)$ and a local maximum at $(2, -3)$.

d.) For the given $f''(x)$, our inflection points may occur at $x = 1$

Intervals	y''	y
$(-\infty, 1)$	$\frac{-}{-} = +$	Concave up
$(1, \infty)$	$\frac{-}{+} = -$	Concave down

So $f(x)$ is concave up on $(-\infty, 1)$ and concave down on $(1, \infty)$ and there are no points of inflection since $f(x)$ is undefined at $x = 1$.

e.) Based on the previous information, the graph of $f(x)$ is shown below:



4. (22 points) The following are unrelated:

a.) (10 points) State the definition of a critical point for a function $f(x)$ and find all critical points of the function $f(x) = |x+2| - 1$

b.) (12 points) If $f(x) = \dfrac{5}{\sqrt{x+2}}$, does the mean value theorem guarantee the existence of a c in (2,7) such that $f'(c) = -1/6$?

Solutions:

a.) A critical point is any point c where either $f'(c) = 0$ or $f'(c)$ does not exist. Writing $f(x)$ as a piecewise function,

$$f(x) = \begin{cases} x+1 & x \geq -2 \\ -x-3 & x < -2 \end{cases}$$

so

$$f'(x) = \begin{cases} 1 & x > -2 \\ -1 & x < -2 \end{cases}$$

thus $f'(x)$ will never equal 0, but $f'(x)$ will not exist at $x = -2$. So we have a critical point at $x = -2$.

b.) $f(x)$ is only discontinuous at $x = -2$, so $f(x)$ is continuous on $[2, 7]$. $f'(x) = \dfrac{-5}{2(x+2)^{3/2}}$, so $f(x)$ is differentiable on $(2, 7)$. Thus, by the mean value theorem, we are guaranteed a c in $(2, 7)$ such that

$$f'(c) = \frac{f(7) - f(2)}{7-2} = \frac{5/\sqrt{9} - 5/\sqrt{4}}{5} = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}.$$

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5. (12 points)

a.) (9 points) Use the linearization of $f(x) = \sqrt[3]{x}$ about $x = 27$ to estimate $\sqrt[3]{24}$

b.) (3 points) Based on your answer above, if $\sqrt[3]{24} \approx 2.8845$, what is the absolute error of your estimate?

Solution:

a.) The linearization of $f(x)$ about $x = a$ is given by

$$L(x) = f(a) + f'(a)(x - a).$$

Here, $a = 27$ and $f(x) = \sqrt[3]{x}$, so $f(a) = 3$ and $f'(x) = \frac{1}{3}x^{-2/3}$ and $f'(27) = \frac{1}{3(27)^{2/3}} = \frac{1}{3(3^2)} = \frac{1}{27}$. Hence,

$$L(x) = 3 + \frac{1}{27}(x - 27)$$

and

$$\sqrt[3]{24} \approx L(24) = 3 + \frac{-3}{27} = 3 - \frac{1}{9} = 2{}^{8}/_{9}.$$

b.) The absolute error is given by

$$|2\frac{8}{9} - 2.8845| \approx |2.8889 - 2.8845| = 0.0044.$$