1. (24 points) Let $f(x) = -2\sqrt{x-2}$ and $g(x) = x^2 - 4$

a.) (4 points) State the domain and range of $f(x)$ using interval notation.

b.) (4 points) State the domain and range of $g(x)$ using interval notation.

c.) (8 points) Calculate $(g \circ f)(x)$ and simplify your answer. State its domain and range using interval notation.

d.) (8 points) Calculate $(f \circ g)(x)$ and simplify your answer. State its domain using interval notation.
Solution: (a) The domain of $f(x)$ is given by $[2, \infty)$. The range is $(-\infty, 0]$.
(b) The domain is $\mathbb{R}$ and and the range is $[-4, \infty)$
(c) We compute

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) = g(-2\sqrt{x-2}) \\ &= (-2\sqrt{x-2})^2 - 4 = 4(x-2) - 4 = 4x - 12.\end{aligned}$$

By definition, $(g \circ f)(x)$ is defined when $f(x)$ is defined and when $f(x)$ is in the domain of $g(x)$. As the domain of $g(x)$ is $\mathbb{R}$, the domain of $(g \circ f)(x)$ is the domain of $f(x)$, or $[2, \infty)$. By plotting $g \circ f(x) = 4x - 12$ on $[2, \infty)$, we see that the range is $[-4, \infty)$.
(d) We compute

$$(f \circ g)(x) = f(x^2 - 4) = -2\sqrt{(x^2-4) - 2} = -2\sqrt{x^2 - 6}.$$

By definition, $(f \circ g)(x)$ is defined when $g(x)$ is defined and when $g(x)$ is in the domain of $f(x)$. The domain of $g(x)$ is $\mathbb{R}$, so we only need to make sure $g(x)$ is in the domain of $f(x)$, or

$$\begin{aligned}x^2 - 4 &\geq 2 \\ x^2 &\geq 6 \\ |x| &\geq \sqrt{6}\end{aligned}$$

So the domain of $(f \circ g)(x)$ is $(-\infty, -\sqrt{6}] \cup [\sqrt{6}, \infty)$.

2. (24 points, 6 points each) Calculate the following limits:

a.) $\lim_{x \to 1} \frac{|2x-2|}{x-1}$

b.) $\lim_{x \to 2} \frac{\cos(\pi x)}{x(x+2)}$

c.) $\lim_{x \to 0} x \sin(1/x)$

d.) $\lim_{x \to \infty} \frac{\sqrt[3]{x^5+4x-2}}{2x^2-9}$

Solution: (a) $\lim_{x \to 1^-} \frac{|2x-2|}{x-1} = \lim_{x \to 1^-} \frac{-(2x-2)}{x-1} = \lim_{x \to 1^-} -2 = -2$,
$\lim_{x \to 1^+} \frac{|2x-2|}{x-1} = \lim_{x \to 1^+} \frac{(2x-2)}{x-1} = \lim_{x \to 1^-} 2 = 2$, So $\lim_{x \to 1^-} \frac{|2x-2|}{x-1} \neq \lim_{x \to 1^+} \frac{|2x-2|}{x-1}$ and the limit does not exist.

(b) By direct substitution, $\lim_{x \to 2} \frac{\cos(\pi x)}{x(x+2)} = \frac{\cos(2\pi)}{2(4)} = \frac{1}{8}$.

(c) Note that $-x \leq x\sin(1/x) \leq x$, and $\lim_{x \to 0} -x = 0 = \lim_{x \to 0} x$. So by the Squeeze Theorem,

$$\lim_{x \to 0} x \sin(1/x) = 0.$$

(d)

$$\begin{aligned}
&\lim_{x \to \infty} \frac{\sqrt[3]{x^5+4x-2}}{2x^2-9} \\
&= \lim_{x \to \infty} \frac{\sqrt[3]{x^5+4x-2}}{2x^2-9} \frac{1/x^2}{1/x^2} \\
&= \lim_{x \to \infty} \frac{\sqrt[3]{x^5+4x-2}/\sqrt[3]{x^6}}{2x^2/x^2 - 9/x^2} \\
&= \lim_{x \to \infty} \frac{\sqrt[3]{x^5+4x-2}/\sqrt[3]{x^6}}{2x^2/x^2 - 9/x^2} \\
&= \lim_{x \to \infty} \frac{\sqrt[3]{x^{-1}+4x^{-5}-2x^{-6}}}{2-9x^{-2}} = 0
\end{aligned}$$

3. (22 points) The following are unrelated:

a.) (8 points) If $f(x)$ is odd and $g(x)$ is even, is $(f \circ g)(x)$ even, odd, or neither? What about $(g \circ f)(x)$?

b.) (6 points) For what values of x is the function $f(x) = \csc(x)$ continuous?

c.) (8 points) Prove that the function $h(x) = \sin(x) - \pi x + \frac{2}{3}x^2$ has at least one zero that is greater than 0.

Solution: (a) $(f \circ g)(-x) = f(g(-x)) = f(g(x)) = (f \circ g)(x)$, so $(f \circ g)(x)$ is even.
$(g \circ f)(-x) = g(f(-x)) = g(-f(x)) = g(f(x)) = (g \circ f)(x)$, so $(g \circ f)(x)$ is even.
(b) Trigonometric functions are continuous on their domains. $\csc(x)$ is defined for $\sin(x) \neq 0$, so $\csc(x)$ is continuous for all x except $x = n\pi$, for n any integer (i.e., $x \neq 0, \pm\pi, \pm 2\pi, \pm 3\pi, ...$).
(c) First note that $h(x)$ is continuous for all x because it is the addition of $\sin(x)$ and a polynomial. Then note that $h(\pi) = \sin(\pi) - \pi^2 + \frac{2\pi^2}{3} = \frac{-\pi^2}{3} < 0$. Then note that $h(2\pi) = \sin(2\pi) - \pi(2\pi) + \frac{2(2\pi)^2}{3} = -2\pi^2 + \frac{8\pi^2}{3} = \frac{2\pi^2}{3} > 0$.
Hence by the Intermediate Value Theorem, there exists a c in the interval $(\pi, 2\pi)$ such that $h(c) = 0$.

4. (20 points) Consider $f(x) = \sqrt{2x+4}$

a.) (3 points) What is the average rate of change of $f(x)$ between $x = 2.5$ and $x = 6$?

b.) (10 points) Use the limit definition of the derivative to calculate $f'(x)$.

c.) (7 points) Find an equation for tangent line to $f(x)$ at $x = 2.5$.
Solution: (a) the difference quotient is given by
(a) The average rate of change between $x = 2.5$ and $x = 6$ is given by

$$\frac{f(6) - f(2.5)}{6 - 2.5} = \frac{\sqrt{16} - \sqrt{9}}{3.5} = \frac{4-3}{3.5} = \frac{1}{3.5} = \frac{2}{7}$$

(b)

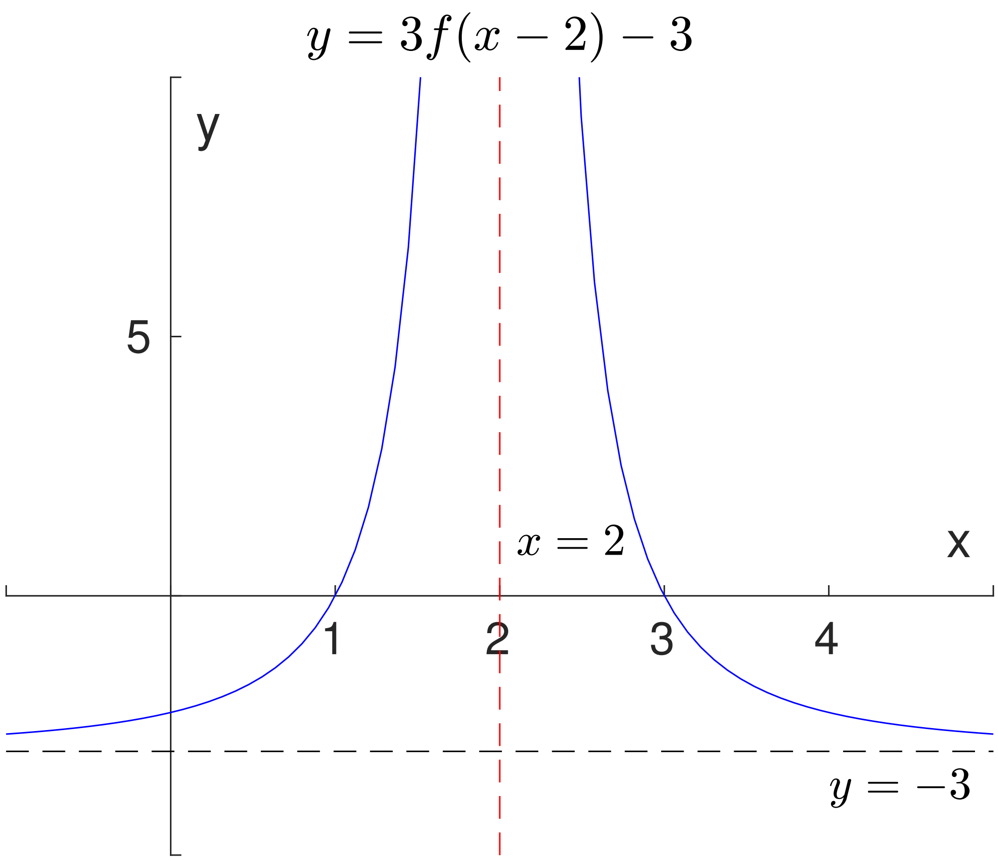
$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{2(x+h)+4} - \sqrt{2x+4}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{2(x+h)+4} - \sqrt{2x+4}}{h} \frac{\sqrt{2(x+h)+4} + \sqrt{2x+4}}{\sqrt{2(x+h)+4} + \sqrt{2x+4}} \\
&= \lim_{h\to 0} \frac{(2(x+h)+4) - (2x+4)}{h(\sqrt{2(x+h)+4} + \sqrt{2x+4})} \\
&= \lim_{h\to 0} \frac{2h}{h(\sqrt{2(x+h)+4} + \sqrt{2x+4})} \\
&= \lim_{h\to 0} \frac{2}{\sqrt{2(x+h)+4} + \sqrt{2x+4}} \\
&= \frac{2}{\sqrt{2x+4} + \sqrt{2x+4}} \\
&= \frac{1}{\sqrt{2x+4}}
\end{aligned}$$

(c) At $x = 2.5$, $f'(2.5) = 1/3$. The line passes through the point $(2.5, f(2.5)) = (2.5, 3)$. Using point-slope form, our line has the form:

$$\begin{aligned}
y - 3 &= \frac{1}{3}(x - 2.5) \\
y &= \frac{x}{3} + 3 - \frac{2.5}{3} = \frac{x}{3} + \frac{13}{6}
\end{aligned}$$

5. (10 points) If $f(x) = \frac{1}{x^2}$, sketch $y = 3f(x-2) - 3$. Clearly label all vertical asymptotes, horizontal asymptotes, and zeros.

Solution: The graph of $y = 3f(x-2) - 3$ is shown in blue below. We have shifted $f(x)$ two units to the right, stretched it vertically by 3, and shifted it 3 units down.



$g(x) = \frac{3}{(x-2)^2} - 3$, so

$$\lim_{x \to \infty} g(x) = -3$$

and

$$\lim_{x \to 2^-} = +\infty,$$

so we have a horizontal asymptote of $y = -3$ and a vertical asymptote of $x = 2$.

The zeros of the function occur when

$$\begin{aligned}\frac{3}{(x-2)^2} - 3 &= 0 \\ \frac{3}{(x-2)^2} &= 3 \\ (x-2)^2 &= 1 \\ (x-2) &= \pm 1 \\ x &= 2 \pm 1\end{aligned}$$

so the zeros occur at $x = 1$ and 3.
