1. (a)(13pts) (i) What is the domain of $g(x) = \dfrac{x^2+x}{x^2+3x+2}$? Give your answer in interval notation. (ii) Find all horizontal asymptotes of $g(x)$, justify your answer with limits.

(b)(13pts) Find the real number a so that the function $f(x) = \begin{cases} \ln(x+e^{x+2}), & \text{if } x > 0 \\ a\cosh(x), & \text{if } x \le 0 \end{cases}$ is continuous for all real numbers. Use limits to answer this question.

(c)(4pts) If $y = \left(\tan^{-1}(x)\right)^2$ then dy/dx is equal to which of the options below? (**No justification necessary** - *Choose only* *one* *answer, copy down the entire answer.*)

(A) $-2\tan^{-3}(x)\sec^2(x)$ (B) $\dfrac{2}{1+x^2}$ (C) $\dfrac{2\tan^{-1}(x)}{1+x^2}$ (D) $2\arctan(x)\text{arcsec}^2(x)$

Solution:

(a)(i)(6pts) Note that the denominator can be factored as $x^2+3x+2 = (x+1)(x+2)$ so we need $x \neq -1$ and $x \neq -2$, that is $\boxed{\text{the domain is } (-\infty,-2)\cup(-2,-1)\cup(-1,\infty).}$

(a)(ii)(7pts) Here, by dominance of powers (or we could also use l'Hospital's Rule), we have

$$\lim_{x\to\infty}\frac{x^2+x}{x^2+3x+2} = \lim_{x\to\infty}\frac{\cancel{x^2}(1+1/x)}{\cancel{x^2}(1+3/x+2/x^2)} = \frac{1+0}{1+0+0} = 1 \text{ and, similarly, } \lim_{x\to-\infty}\frac{x^2+x}{x^2+3x+2} = 1$$

thus $\boxed{\text{the horizontal asymptote of } g(x) \text{ is } y = 1.}$

(b)(13pts) Recall that continuity requires $\lim_{x\to a} f(x) = f(a)$ for all real numbers a in the domain of $f(x)$ and note that $f(x)$ is continuous for $x \neq 0$ by properties of continuous functions. At $x = 0$, we have

$$\lim_{x\to0^+} f(x) = \lim_{x\to0^+}\ln(x+e^{x+2}) = \ln(e^2) = 2 \text{ and } \lim_{x\to0^-} f(x) = \lim_{x\to0^-} a\cosh(x) = a\cdot\cosh(0) = a\cdot 1 = a = f(0)$$

thus $\boxed{\text{for } f(x) \text{ to be continuous for all real numbers we need } a = 2.}$

(c)(4pts) $\boxed{\text{Choice C.}}$ By the Chain Rule we have,

$$\frac{d}{dx}[(\tan^{-1}(x))^2] = 2\cdot\tan^{-1}(x)\cdot\frac{1}{1+x^2} = \frac{2\tan^{-1}(x)}{1+x^2} \Rightarrow \text{ Choice C.}$$

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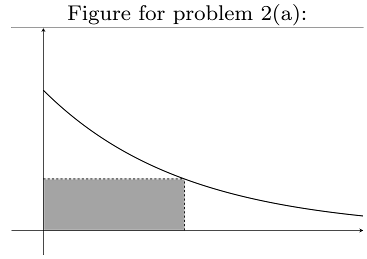
2. (a)(15pts) What is the area of the largest rectangle in the first quadrant with two sides on the axes and one corner on the curve $y = e^{-x}$? Show all work and be sure to classify your answer either using the First Derivative Test or the Second Derivative Test.

Figure for problem 2(a):



(b)(15pts) Evaluate the definite integral $\displaystyle\int_1^{\sqrt{3}} \frac{6}{1+x^2}\,dx$. Simplify your answer.

(c)(5pts) If $f(x) = \dfrac{\text{sech}^2(x)}{2+\tanh(x)}$, then which of the choices below corresponds to $\int f(x)\,dx$? (**No justification necessary** - *Choose only* *one* *answer, copy down the entire answer.*)

(A) $\dfrac{(2+\tanh(x))^2}{2}+C$ (B) $\ln(\tanh(x))+C$ (C) $\dfrac{2\text{sech}(x)\tanh(x)}{2+\tanh(x)}+C$ (D) $\ln|2+\tanh(x)|+C$

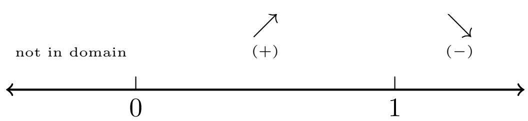
Solution:

(a)(15pts) Note that we wish to maximize the area of the rectangle which is $A = xy$ where $y = e^{-x}$ and $x \geq 0$. Thus we have

$$A = xy \Rightarrow A(x) = xe^{-x} \Rightarrow A'(x) = e^{-x} - xe^{-x} = e^{-x}(1-x) \text{ and } A'(x) = 0 \Rightarrow x = 1$$

and so $x = 1$ is a critical point. Now, using the First Derivative Test, we have that

Sign chart for $f'(x)$



so we have a local max at $x = 1$ which also is an absolute max thus $\boxed{\text{the area of the largest rectangle is } A(1) = 1 \cdot e^{-1} = 1/e}$ (note that when $x = 0$ we have an absolute minimum area of $A = 0$).

(b)(15pts) Here we have

$$\int_1^{\sqrt{3}} \frac{6}{1+x^2}\,dx = 6\int_1^{\sqrt{3}} \frac{1}{1+x^2}\,dx = 6\tan^{-1}(x)\Big|_1^{\sqrt{3}} = 6\left(\tan^{-1}(\sqrt{3}) - \tan^{-1}(1)\right) = 6\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = 6 \cdot \frac{\pi}{12} = \boxed{\pi/2}$$

(c)(5pts) $\boxed{\text{Choice D.}}$ Note that if we let $u = 2 + \tanh(x)$ then $du = \text{sech}^2(x)dx$ thus

$$\int \frac{\text{sech}^2(x)}{2+\tanh(x)}\,dx = \int \frac{1}{u}du = \ln|u| + C = \ln|2+\tanh(x)| + C \Rightarrow \text{Choice D.}$$

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3. (a)(13pts) Use logarithmic differentiation to find the derivative of $y = (\sec x)^{\ln x}$. Show all work.

(b)(13pts) Find the linearization of $f(x) = \displaystyle\int_0^{\sin(x)} \sqrt{1+t^2}\,dt$ centered at the point $a = \pi$. Show all work.

(c)(4pts) The definite integral $\displaystyle\int_0^1 xe^{-2x^2}\,dx$ is equal to which choice below? (**No justification necessary** - *Choose only one answer, copy down the entire answer.*)

(A) $\frac{1}{4}(1-e)$ (B) $\frac{1}{4}(e^{-2}-1)$ (C) $\dfrac{1-e^{-2}}{4}$ (D) $-e^{-1}-e$

Solution:

(a)(13pts) Taking the natural log of both sides yields

$$y = (\sec x)^{\ln x} \Rightarrow \ln(y) = \ln\left[(\sec x)^{\ln x}\right] \Rightarrow \ln(y) = \ln(x)\ln(\sec x)$$

and differentiation yields

$$\frac{y'}{y} = \frac{1}{x} \cdot \ln(\sec x) + \ln(x) \cdot \frac{1}{\sec(x)} \cdot \sec(x)\tan(x) = \frac{\ln(\sec x)}{x} + \ln(x)\tan(x)$$

thus we have

$$y' = y\left[\frac{\ln(\sec x)}{x} + \ln(x)\tan(x)\right] \Rightarrow \boxed{y' = (\sec x)^{\ln x}\left[\frac{\ln(\sec x)}{x} + \ln(x)\tan(x)\right]}$$

(b)(13pts) The linearization is $L(x) = f(\pi) + f'(\pi)(x-\pi)$ where $f(\pi) = \int_0^{\sin(\pi)} \sqrt{1+t^2}\,dt = \int_0^0 \sqrt{1+t^2}\,dt = 0$ and

$$f'(x) = \frac{d}{dx}\left[\int_0^{\sin(x)} \sqrt{1+t^2}\,dt\right] = \sqrt{1+(\sin x)^2}\cdot\cos x \text{ thus } f'(\pi) = \sqrt{1+(\sin\pi)^2}\cdot\cos\pi = \sqrt{1+0^2}\cdot(-1) = -1$$

thus the linearization is $L(x) = f(\pi) + f'(\pi)(x-\pi) = 0 + (-1)(x-\pi) \Rightarrow \boxed{L(x) = \pi - x.}$

(c)(4pts) $\boxed{\text{Choice C.}}$ Note that if we let $u = -2x^2$ then $du = -4xdx$ and so $-\frac{du}{4} = xdx$ thus we have

$$\int_0^1 xe^{-2x^2}\,dx = -\frac{1}{4}\int_0^{-2} e^u\,du = -\frac{1}{4}e^u\Big|_0^{-2} = -\frac{1}{4}\left(e^{-2} - e^0\right) = \frac{1-e^{-2}}{4} \Rightarrow \text{Choice C.}$$

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4. (a)(15pts) Use l'Hospital's Rule to evaluate the limit $\lim_{x\to\infty} x\tan(1/x)$. Show all work.

(b)(15pts) The *half-life* of Cesium-137 is 30 years. Suppose we initially have a 100-mg sample. (i) Find a formula for the mass remaining after t years. (ii) Set-up (**but do not evaluate**) an integral to calculate the *average value* of the mass remaining of Cesium-137 after 10 years.

(c)(5pts) Which graph below most closely resembles the graph of $g(x) = \frac{x^2+x}{x^2+3x+2}$? (**No justification necessary** - *Choose only* <u>*one*</u> *answer.*)

Solution:

(a)(15pts) Here we have a "$0\cdot\infty$" type indeterminate form, thus

$$\lim_{x\to\infty} x\tan(1/x) = \lim_{x\to\infty}\frac{\tan(1/x)}{1/x} \stackrel{L'H}{=} \lim_{x\to\infty}\frac{\sec^2(1/x)\cdot -1/x^2}{-1/x^2} = \sec^2(0) = \boxed{1}$$

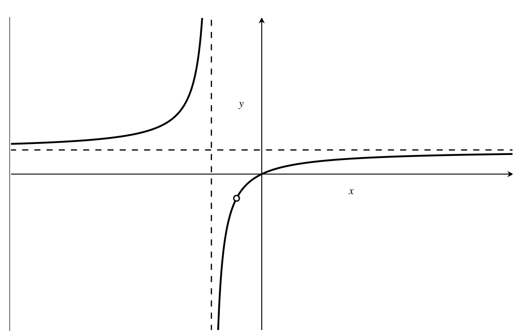
(b)(i)(8pts) Note that $y(t) = y_0e^{kt} = 100e^{kt}$ and since the half-life is 30 years, we have

$$\frac{100}{2} = 100e^{k\cdot 30} \Rightarrow \frac{1}{2} = e^{k\cdot 30} \Rightarrow \ln\left(\frac{1}{2}\right) = 30k \Rightarrow k = \frac{\ln(1/2)}{30} = -\frac{\ln(2)}{30} \Rightarrow \boxed{y(t) = 100e^{-\ln(2)t/30}}$$

(b)(ii)(7pts) The average value of the mass remaining of Cesium-137 after 10 years is

$$\boxed{f_{ave} = \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{10}\int_0^{10} 100e^{-\ln(2)t/30}\,dt}$$

(c)(5pts) $\boxed{\text{Choice C.}}$ Note that $g(x) = \frac{x^2+x}{x^2+3x+2} = \frac{x(x+1)}{(x+1)(x+2)} = \frac{x}{x+2}$ if $x \neq 1$. Thus we have a vertical asymptote at $x = -2$ and since $\lim_{x\to -1} g(x) = -1$ there is a removable discontinuity at $x = -1$, that is, there is a hole at $(-1,-1)$. Also, recall from problem 1(a)(ii), there is a horizontal asymptote at $y = 1$ since $\lim_{x\to\pm\infty} g(x) = 1$. Thus the graph of $g(x)$ most resembles choice C:



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5. (20pts) Answer either **ALWAYS TRUE** or **FALSE**. You do NOT need to justify your answer. (*Don't just write down "A.T." or "F", completely write out the words "ALWAYS TRUE" or "FALSE" depending on your answer.*)

(a)(5 pts) If the velocity of a particle at time t seconds is $v(t) = 2t - 1$ meters per second, then the *total distance* traveled during the time period $0 \le t \le 1$ by the particle is 0.25 meters.

(b)(5 pts) By the Intermediate Value Theorem, the equation $\log_2(x) + x = 0$, for $0.5 \le x \le 4$, has at least one root in the interval $(0.5, 4)$.

(c)(5 pts) If the function $f(x)$ is continuous for all real values of x then $f(x)$ is differentiable for all real values of x.

(d)(5 pts) If $f(x) = \ln(x) + \tan^{-1}(x)$ then $f(1) = \pi/4$ and $(f^{-1})'(\pi/4) = 2/3$.

Solution: (a) F (b) A.T. (c) F (d) A.T.

Discussion:

(a)(5pt) The total distance is

$$\int_0^1 |2t-1|\,dt = \int_0^{1/2} -(2t-1)\,dt + \int_{1/2}^1 (2t-1)\,dt = \left(t - t^2\right)\Big|_0^{1/2} + \left(t^2 - t\right)\Big|_{1/2}^1$$

$$= \left[\left(\frac{1}{2} - \frac{1}{4}\right) - 0\right] + \left[0 - \left(\frac{1}{4} - \frac{1}{2}\right)\right] = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \neq 0.25 \Rightarrow \boxed{\text{F}}$$

(b)(5pt) First note that

$$f(0.5) = \log_2(1/2) + \frac{1}{2} = \log_2(2^{-1}) + \frac{1}{2} = -1 + \frac{1}{2} = -\frac{1}{2} < 0$$

and

$$f(4) = \log_2(4) + 4 = \log_2(2^2) + 4 = 2 + 4 = 6 > 0$$

and also notice that $f(x) = \log_2(x) + x$ is continuous for $x > 0$ thus by the Intermediate Value Theorem there exists at least one number c in $(0.5, 4)$ such that $f(c) = 0$, that is, the equation $\log_2(x) + x = 0$, for $0.5 \le x \le 4$, has at least one root in the interval $(0.5, 4) \Rightarrow \boxed{\text{A.T.}}$

(c)(5pt) The function $f(x) = |x|$ is one counterexample to this statement. Note that $f(x) = |x|$ is continuous for all real values of x but $f(x) = |x|$ is **not** differentiable at $x = 0 \Rightarrow \boxed{\text{F}}$.

(d)(5pt) First note that the domain of $f(x) = \ln(x) + \tan^{-1}(x)$ is $x > 0$ and thus $f'(x) = \dfrac{1}{x} + \dfrac{1}{1+x^2} > 0$ since $x > 0$ and so $f(x)$ is an increasing function and therefore passes the horizontal line test and so has an inverse. Now note that

$$f(1) = \ln(1) + \tan^{-1}(1) = 0 + \frac{\pi}{4} \text{ and so } f(1) = \pi/4 \Rightarrow 1 = f^{-1}(\pi/4)$$

and according to a theorem from Chapter 5 (see section 5.1) we have

$$\frac{d}{dx}\left(f^{-1}(\pi/4)\right) = \frac{1}{f'(f^{-1}(\pi/4))} = \frac{1}{f'(1)} = \frac{1}{1 + 1/2} = \frac{1}{3/2} = 2/3 \Rightarrow \boxed{\text{A.T.}}$$