1. (a)(13pts) (i) What is the domain of \( g(x) = \frac{x^2 + x}{x^2 + 3x + 2} \)? Give your answer in interval notation. (ii) Find all horizontal asymptotes of \( g(x) \), justify your answer with limits.

(b)(13pts) Find the real number \( a \) so that the function \( f(x) = \begin{cases} \ln(x + e^{x+2}), & \text{if } x > 0 \\ a \cosh(x), & \text{if } x \leq 0 \end{cases} \) is continuous for all real numbers. Use limits to answer this question.

(c)(4pts) If \( y = \tan^{-1}(x)^2 \) then \( dy/dx \) is equal to which of the options below? (No justification necessary - Choose only one answer, copy down the entire answer.)

A: \(-2 \tan^{-3}(x) \sec^2(x)\)   
B: \(\frac{2}{1 + x^2}\)   
C: \(2 \tan^{-1}(x) \)   
D: \(2 \arctan(x) \text{arcsec}^2(x)\)

Solution:

(a)(i)(6pts) Note that the denominator can be factored as \( x^2 + 3x + 2 = (x + 1)(x + 2) \) so we need \( x \neq -1 \) and \( x \neq -2 \), that is the domain is \((-\infty, -2) \cup (-2, -1) \cup (-1, \infty)\).

(a)(ii)(7pts) Here, by dominance of powers (or we could also use l’Hospital’s Rule), we have \( \lim_{x \to \infty} x^2 + x = \lim_{x \to \infty} x^2(1 + 1/x^2) = 1 + 0 = 1 \) and, similarly, \( \lim_{x \to -\infty} x^2 + x = 1 \) thus the horizontal asymptote of \( g(x) \) is \( y = 1 \).

(b)(13pts) Recall that continuity requires \( \lim_{x \to a} f(x) = f(a) \) for all real numbers \( a \) in the domain of \( f(x) \) and note that \( f(x) \) is continuous for \( x \neq 0 \) by properties of continuous functions. At \( x = 0 \), we have

\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \ln(x + e^{x+2}) = \ln(e^2) = 2 \quad \text{and} \quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} a \cosh(x) = a \cdot \cosh(0) = a \cdot 1 = a = f(0)
\]

thus for \( f(x) \) to be continuous for all real numbers we need \( a = 2 \).

(c)(4pts) Choice C. By the Chain Rule we have,

\[
\frac{d}{dx} \left[ (\tan^{-1}(x))^2 \right] = 2 \cdot \tan^{-1}(x) \cdot \frac{1}{1 + x^2} = \frac{2 \tan^{-1}(x)}{1 + x^2} \Rightarrow \text{Choice C.}
\]

2. (a)(15pts) What is the area of the largest rectangle in the first quadrant with two sides on the axes and one corner on the curve \( y = e^{-x} \)? Show all work and be sure to classify your answer either using the First Derivative Test or the Second Derivative Test.

(b)(15pts) Evaluate the definite integral \( \int_{1}^{\sqrt{3}} \frac{6}{1 + x^2} \, dx \). Simplify your answer.

(c)(5pts) If \( f(x) = \frac{\text{sech}^2(x)}{2 + \text{tanh}(x)} \), then which of the choices below corresponds to \( \int f(x) \, dx \)? (No justification necessary - Choose only one answer, copy down the entire answer.)
\[
\frac{(2 + \tanh(x))^2}{2} + C \quad \ln(\tanh(x)) + C \quad \frac{2\text{sech}(x)\tanh(x)}{2 + \tanh(x)} + C \quad \ln|2 + \tanh(x)| + C
\]

**Solution:**

(a)(15pts) Note that we wish to maximize the area of the rectangle which is \(A = xy\) where \(y = e^{-x}\) and \(x \geq 0\). Thus we have

\[A = xy \Rightarrow A(x) = xe^{-x} \Rightarrow A'(x) = e^{-x} - xe^{-x} = e^{-x}(1 - x)\]

and so \(x = 1\) is a critical point. Now, using the First Derivative Test, we have that

\[
\text{Sign chart for } f'(x)
\]

\[
\begin{array}{c|c|c}
 & (+) & (-) \\
\hline
0 & \nearrow & \searrow \\
\end{array}
\]

so we have a local max at \(x = 1\) which also is an absolute max thus the area of the largest rectangle is \(A(1) = 1 \cdot e^{-1} = 1/e\) (note that when \(x = 0\) we have an absolute minimum area of \(A = 0\)).

(b)(15pts) Here we have

\[
\int_{\sqrt{3}}^{1} \frac{6}{1 + x^2} \, dx = 6 \left[ \int_{1}^{\sqrt{3}} \frac{1}{1 + x^2} \, dx \right] = 6 \left[ \tan^{-1}(\sqrt{3}) - \tan^{-1}(1) \right] = 6 \left( \frac{\pi}{3} - \frac{\pi}{4} \right) = 6 \cdot \frac{\pi}{12} = \frac{\pi}{2}
\]

(c)(5pts) [Choice D.] Note that if we let \(u = 2 + \tanh(x)\) then \(du = \text{sech}^2(x) \, dx\) thus

\[
\int \frac{\text{sech}^2(x)}{2 + \tanh(x)} \, dx = \int \frac{1}{u} \, du = \ln|u| + C = \ln|2 + \tanh(x)| + C \Rightarrow \text{Choice D.}
\]

3. (a)(13pts) Use logarithmic differentiation to find the derivative of \(y = (\sec x)^{\ln x}\). Show all work.

(b)(13pts) Find the linearization of \(f(x) = \int_{0}^{\sin(x)} \sqrt{1 + t^2} \, dt\) centered at the point \(a = \pi\). Show all work.

(c)(4pts) The definite integral \(\int_{0}^{1} xe^{-2x^2} \, dx\) is equal to which choice below? (No justification necessary - Choose only one answer, copy down the entire answer.)

\[
(A) \frac{1}{4}(1 - e) \quad \quad (B) \frac{1}{4}(e^{-2} - 1) \quad \quad (C) \frac{1 - e^{-2}}{4} \quad \quad (D) -e^{-1} - e
\]

**Solution:**

(a)(13pts) Taking the natural log of both sides yields

\[y = (\sec x)^{\ln x} \Rightarrow \ln(y) = \ln[(\sec x)^{\ln x}] \Rightarrow \ln(y) = \ln(x) \ln(\sec x)\]

and differentiation yields

\[
\frac{y'}{y} = \frac{1}{x} \cdot \ln(\sec x) + \ln(x) \cdot \frac{1}{\sec(x)} \cdot \sec(x) \tan(x) = \frac{\ln(\sec x)}{x} + \ln(x) \tan(x)
\]

thus we have

\[
y' = y \left[ \frac{\ln(\sec x)}{x} + \ln(x) \tan(x) \right] \Rightarrow y = (\sec x)^{\ln x} \left[ \frac{\ln(\sec x)}{x} + \ln(x) \tan(x) \right]
\]
(b)(13pts) The linearization is \( L(x) = f(\pi) + f'(\pi)(x - \pi) \) where \( f(\pi) = \int_0^{\sin(\pi)} \sqrt{1 + t^2} \, dt = \int_0^0 \sqrt{1 + t^2} \, dt = 0 \) and
\[
f'(x) = \frac{d}{dx} \left[ \int_0^{\sin(x)} \sqrt{1 + t^2} \, dt \right] = \sqrt{1 + (\sin x)^2} \cdot \cos x \quad \text{thus} \quad f'(\pi) = \sqrt{1 + (\sin \pi)^2} \cdot \cos \pi = \sqrt{1 + 0^2} \cdot (-1) = -1
\]
thus the linearization is \( L(x) = f(\pi) + f'(\pi)(x - \pi) = 0 + (-1)(x - \pi) \) \( \Rightarrow L(x) = \pi - x \).

(c)(4pts) \( \text{Choice C} \). Note that if we let \( u = -2x^2 \) then \( du = -4xdx \) and so \( -\frac{du}{4} = xdx \) thus we have
\[
\int_0^1 xe^{-2x^2} \, dx = -\frac{1}{4} \int_0^{-2} e^u \, du = -\frac{1}{4} e^u \bigg|_0^{\pi} = \left( -\frac{1}{4} (e^2 - e^0) \right) = \frac{1 - e^{-2}}{4} \Rightarrow \text{Choice C}.
\]

4. (a)(15pts) Use l’Hospital’s Rule to evaluate the limit \( \lim_{x \to \infty} x \tan(1/x) \). Show all work.

(b)(15pts) The half-life of Cesium-137 is 30 years. Suppose we initially have a 100-mg sample. (i) Find a formula for the mass remaining after \( t \) years. (ii) Set-up (but do not evaluate) an integral to calculate the average value of the mass remaining of Cesium-137 after 10 years.

(c)(5pts) Which graph below most closely resembles the graph of \( g(x) = \frac{x^2 + x}{x^2 + 3x + 2} \) (No justification necessary - Choose only one answer.)

Solution:

(a)(15pts) Here we have a “0 · \infty” type indeterminate form, thus
\[
\lim_{x \to \infty} x \tan(1/x) = \lim_{x \to \infty} \frac{\tan(1/x)}{1/x} \overset{L’H}{=} \lim_{x \to \infty} \frac{\sec^2(1/x) \cdot (-1/x^2)}{1/x^2} = \sec^2(0) = 1
\]

(b)(i)(8pts) Note that \( y(t) = y_0 e^{kt} = 100e^{kt} \) and since the half-life is 30 years, we have
\[
\frac{100}{2} = 100e^{k \cdot 30} \Rightarrow \frac{1}{2} = e^{k \cdot 30} \Rightarrow \ln \left( \frac{1}{2} \right) = 30k \Rightarrow k = \frac{\ln(1/2)}{30} = -\frac{\ln(2)}{30} \Rightarrow y(t) = 100e^{-\ln(2)t/30}
\]

(b)(ii)(7pts) The average value of the mass remaining of Cesium-137 after 10 years is
\[
f_{ave} = \frac{1}{b - a} \int_a^b f(t) \, dt = \frac{1}{10} \int_0^{10} 100e^{-1.5t/30} \, dt
\]

(c)(5pts) \( \text{Choice C} \). Note that \( g(x) = \frac{x^2 + x}{x^2 + 3x + 2} = \frac{x(x + 1)}{(x + 1)(x + 2)} = \frac{x}{x + 2} \) if \( x \neq 1 \). Thus we have a vertical asymptote at \( x = -2 \) and since \( \lim_{x \to -1} g(x) = -1 \) there is a removable discontinuity at \( x = -1 \), that is, there is a hole at \((-1, -1)\). Also, recall from problem 1(a)(ii), there is a horizontal asymptote at \( y = 1 \) since \( \lim_{x \to \pm\infty} g(x) = 1 \). Thus the graph of \( g(x) \) most resembles choice C:

![Graph of g(x)](attachment:graph.png)
5. (20pts) Answer either **ALWAYS TRUE** or **FALSE**. You do **NOT** need to justify your answer. (Don’t just write down “A.T.” or “F”, completely write out the words “ALWAYS TRUE” or “FALSE” depending on your answer.)

(a) (5 pts) If the velocity of a particle at time \( t \) seconds is \( v(t) = 2t - 1 \) meters per second, then the total distance traveled during the time period \( 0 \leq t \leq 1 \) by the particle is \( 0.25 \) meters.

(b) (5 pts) By the Intermediate Value Theorem, the equation \( \log_2(x) + x = 0 \), for \( 0.5 \leq x \leq 4 \), has at least one root in the interval \((0.5, 4)\).

(c) (5 pts) If the function \( f(x) \) is continuous for all real values of \( x \) then \( f(x) \) is differentiable for all real values of \( x \).

(d) (5 pts) If \( f(x) = \ln(x) + \tan^{-1}(x) \) then \( f(1) = \pi/4 \) and \( (f^{-1})'(\pi/4) = 2/3 \).

**Solution:**

(a) F  (b) A.T.  (c) F  (d) A.T.

**Discussion:**

(a) (5pt) The total distance is

\[
\int_0^1 |2t - 1| \, dt = \int_0^{1/2} -(2t - 1) \, dt + \int_{1/2}^1 (2t - 1) \, dt = \left[ -t + t^2 \right]_0^{1/2} + \left[ t - t^2 \right]_{1/2}^1 = \left[ \left( \frac{1}{2} - \frac{1}{2} \right) - 0 \right] + \left[ 0 - \left( \frac{1}{4} - \frac{1}{4} \right) \right] = \frac{1}{4} + \frac{1}{4} = 1/2 \neq 0.25 \Rightarrow F
\]

(b) (5pt) First note that

\[
f(0.5) = \log_2(1/2) + \frac{1}{2} = \log_2(2^{-1}) + \frac{1}{2} = -1 + \frac{1}{2} = -1/2 < 0
\]

and

\[
f(4) = \log_2(4) + 4 = \log_2(2^2) + 4 = 2 + 4 = 6 > 0
\]

and also notice that \( f(x) = \log_2(x) + x \) is continuous for \( x > 0 \) thus by the Intermediate Value Theorem there exists at least one number \( c \) in \((0.5, 4)\) such that \( f(c) = 0 \), that is, the equation \( \log_2(x) + x = 0 \), for \( 0.5 \leq x \leq 4 \), has at least one root in the interval \((0.5, 4)\) \( \Rightarrow A.T. \)

(c) (5pt) The function \( f(x) = |x| \) is one counterexample to this statement. Note that \( f(x) = |x| \) is continuous for all real values of \( x \) but \( f(x) = |x| \) is **not** differentiable at \( x = 0 \) \( \Rightarrow F \)

(d) (5pt) First note that the domain of \( f(x) = \ln(x) + \tan^{-1}(x) \) is \( x > 0 \) and thus \( f'(x) = \frac{1}{x} + \frac{1}{1 + x^2} > 0 \) since \( x > 0 \) and so \( f(x) \) is an increasing function and therefore passes the horizontal line test and so has an inverse. Now note that

\[
f(1) = \ln(1) + \tan^{-1}(1) = 0 + \frac{\pi}{4} \text{ and so } f(1) = \pi/4 \Rightarrow 1 = f^{-1}(\pi/4)
\]

and according to a theorem from Chapter 5 (see section 5.1) we have

\[
\frac{d}{dx} (f^{-1}(\pi/4)) = \frac{1}{f'(f^{-1}(\pi/4))} = \frac{1}{f'(1)} = \frac{1}{1 + 1/2} = \frac{1}{3/2} = 2/3 \Rightarrow A.T.
\]