1. The following problems are not related.

(a)(11pts) Suppose the acceleration of an object at any time $t$ is given by $a(t) = 3t^2 - 4t$ m/s$^2$, $t \geq 0$. Find the velocity, $v(t)$, at any time $t$ if $v(1) = 1$ m/s. Show all work.

(b)(11pts) Each of the regions $A$, $B$ and $C$ bounded by the graph of $f(x)$ and the $x$-axis has an area of 5, see below. Find the value of $\int_0^2 \left[ \pi^2 f(x) - 4 \right] \, dx$.

(c)(4pts) Using right endpoints and subintervals of equal width, which of the limits below is equal to $\int_\pi^{2\pi} \sin(x) \, dx$?

(No justification necessary - Choose only one answer, copy down the entire answer.)

(A) $\lim_{n \to \infty} \sum_{i=1}^{n} \sin \left( \frac{i\pi}{n} \right)$  
(B) $\lim_{n \to \infty} \sum_{i=1}^{n} \sin \left( \frac{\pi i}{n} \right)$  
(C) $\lim_{n \to \infty} \sum_{i=1}^{n} \frac{\pi \sin(\pi i/n)}{n}$  
(D) $\lim_{n \to \infty} \sum_{i=1}^{n} \frac{\pi \sin(\pi + i\pi/n)}{n}$

Solution:

(a)(11pts) Recall that $v(t) = \int a(t) \, dt = \int (3t^2 - 4t) \, dt = t^3 - 4 \cdot \frac{t^2}{2} + C = t^3 - 2t^2 + C$

and $v(1) = 1$ implies $1 = v(1) = 1^3 - 2 \cdot 1^2 + C \Rightarrow 1 = -1 + C \Rightarrow C = 2 \Rightarrow v(t) = t^3 - 2t^2 + 2$

(b)(11pts) First note that $\int_0^2 f(x) \, dx = -5$ thus

$\int_2^0 \left[ \pi^2 f(x) - 4 \right] \, dx = -\int_0^2 \left[ \pi^2 f(x) - 4 \right] \, dx = -\pi^2 \int_0^2 f(x) \, dx + \int_0^2 4 \, dx = -\pi^2 \cdot (-5) + 4 \cdot (2 - 0) = 8 + 5\pi^2$

(c)(4pts) Choice D. Note that $\int_a^b f(x) \, dx = \int_\pi^{2\pi} \sin(x) \, dx$ and $\Delta x = \frac{b - a}{n} = \frac{2\pi - \pi}{n} = \frac{\pi}{n}$ and $x_i = a + i\Delta x = \pi + i(\pi/n)$ thus

$\int_\pi^{2\pi} \sin(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} \sin(x_i) \Delta x = \lim_{n \to \infty} \sum_{i=1}^{n} \sin \left( \pi + i\pi/n \right) \frac{\pi}{n}$

2. The following problems are not related.

(a)(11pts) Suppose we want to approximate a solution to the equation $3\sin(x) = x$ using Newton’s Method. What would the formula for $x_{n+1}$ be? (To get full points for this question you must provide the explicit formula for $x_{n+1}$ in terms of $x_n$, the generic formula for Newton’s Method is not sufficient. You do not need to approximate the solution.)
(b)(11pts) Write the expression \( \int_{-2}^{5} f(x) \, dx + \int_{-2}^{2} f(t) \, dt - \int_{-2}^{-1} f(x) \, dx \) as a single integral in the form \( \int_{a}^{b} f(x) \, dx \).

(c)(4pts) Using differentiation, which one of the choices below can be verified to be equivalent to \( \int x \cos(x) \, dx \)? (No justification necessary. Choose only one answer, copy down the entire answer.)

(A) \( \frac{x^2}{2} \sin(x) + C \)  
(B) \( \cos(x) + x \sin(x) + C \)  
(C) \( x \sin(x) + C \)  
(D) \( \frac{\cos^2(x)}{2} + C \)  
(E) \( \sin(x) + C \)

Solution:

(a)(11pts) Note that \( 3 \sin(x) = x \Rightarrow 3 \sin(x) - x = 0 \), thus if we let \( f(x) = 3 \sin(x) - x \) then

\[
\begin{align*}
x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\
&= x_n - 3 \frac{\sin(x_n) - x_n}{\cos(x_n) - 1} \\
&= \frac{3x_n \cos(x_n) - 3 \sin(x_n) - x_n}{\cos(x_n) - 1} = \frac{3[x_n \cos(x_n) - \sin(x_n)]}{\cos(x_n) - 1}
\end{align*}
\]

(b)(11pts) Note that in a definite integral the variable of integration is a dummy variable and can be replaced with any other variable so we can write \( \int_{-2}^{5} f(t) \, dt = \int_{-2}^{2} f(x) \, dx \) thus

\[
\int_{-2}^{5} f(x) \, dx + \int_{-2}^{2} f(t) \, dt - \int_{-2}^{-1} f(x) \, dx = \int_{-1}^{-2} f(x) \, dx + \int_{2}^{5} f(x) \, dx + \int_{-1}^{2} f(x) \, dx = \int_{-1}^{5} f(x) \, dx
\]

(c)(4pts) Choice B. Note that

\[
\frac{d}{dx} [\cos(x) + x \sin(x) + C] = -\sin(x) + \sin(x) + x \cos(x) + 0 = x \cos(x) \Rightarrow \int x \cos(x) \, dx = \cos(x) + x \sin(x) + C
\]

3. The following problems are not related. Show all work.

(a)(12pts) Use the Fundamental Theorem of Calculus to evaluate the definite integral \( \int_{-\pi}^{\pi} |3 \sin(x)| \, dx \).

(b)(12pts) If \( f(x) = \int_{4}^{x} \frac{t - 1}{t^2 + 1} \, dt \), use the Fundamental Theorem of Calculus to find \( f'(2) \). Simplify your answer.

Solution:

(a)(12pts) First note that

\[
\int_{-\pi}^{\pi} |3 \sin(x)| \, dx = \int_{-\pi}^{0} -3 \sin(x) \, dx + \int_{0}^{\pi} 3 \sin(x) \, dx
\]

Now, by the Fundamental Theorem of Calculus, we have

\[
\int_{-\pi}^{0} -3 \sin(x) \, dx + \int_{0}^{\pi} 3 \sin(x) \, dx = -3 \cdot \cos(x) \bigg|_{-\pi}^{0} + 3 \cdot \cos(x) \bigg|_{0}^{\pi}
\]

\[
= 3[\cos(0) - \cos(-\pi)] + 3[-\cos(\pi) + \cos(0)]
\]

\[
= 3[1 - (-1)] + 3[-(-1) + 1] = 6 + 6 = 12
\]

(alternately, note that \( f(x) = |3 \sin(x)| \) is an even function thus \( \int_{-\pi}^{\pi} |3 \sin(x)| \, dx = 2 \int_{0}^{\pi} 3 \sin(x) \, dx = 2 \cdot 6 = 12 \).)

(b)(12pts) By the Fundamental Theorem of Calculus we have

\[
f'(x) = \frac{d}{dx} \left[ \int_{4}^{x^2} \frac{t - 1}{t^2 + 1} \, dt \right] = \frac{x^2 - 1}{(x^2)^2 + 1} \cdot 2x = \frac{2x(x^2 - 1)}{x^4 + 1} \Rightarrow f'(2) = \frac{2 \cdot 2 \cdot (2^2 - 1)}{2^4 + 1} = \frac{4 \cdot 3}{17} = \frac{12}{17}
\]
4. The following problems are not related. Show all work.

(a)(12pts) Use $u$-substitution to evaluate the definite integral $\int_0^4 \frac{\sqrt{1 + \sqrt{x}}}{\sqrt{x}} \, dx$. Show all work.

(b)(12pts) Find the average value of $f(x) = \frac{\sin(x)}{\sec(x)}$ on the interval $[0, \pi/4]$. Justify your answer.

Solution:

(a)(12pts) If we let $u = 1 + \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}} \, dx \Rightarrow 2 \, du = \frac{1}{\sqrt{x}} \, dx$ also note that $x = 0 \Rightarrow u = 1 + \sqrt{0} = 1$ and $x = 4 \Rightarrow u = 1 + \sqrt{4} = 3$ thus

$$\int_0^4 \frac{\sqrt{1 + \sqrt{x}}}{\sqrt{x}} \, dx = \int_0^4 \left(1 + \sqrt{x}\right)^{1/3} \cdot \frac{1}{\sqrt{x}} \, dx = \int_1^3 u^{1/3} \cdot 2 \, du = 2 \cdot \frac{3}{4} u^{4/3} \bigg|_1^3 = \frac{3}{2} \left(3^{4/3} - 1\right)$$

(b)(12pts) Recall that the average value of $f(x)$ over the interval $[a, b]$ is given by

$$f_{\text{ave}} = \frac{1}{b - a} \int_a^b f(x) \, dx = \frac{1}{\pi/4 - 0} \int_0^{\pi/4} \frac{\sin(x)}{\sec(x)} \, dx = \frac{4}{\pi} \int_0^{\pi/4} \sin(x) \cos(x) \, dx$$

now if we let $u = \sin(x) \Rightarrow du = \cos(x) \, dx$ and $x = 0 \Rightarrow u = \sin(0) = 0$ and $x = \pi/4 \Rightarrow u = \sin(\pi/4) = \sqrt{2}/2$ thus

$$\frac{4}{\pi} \int_0^{\pi/4} \sin(x) \cos(x) \, dx = \frac{4}{\pi} \int_0^{\sqrt{2}/2} u \, du = \frac{4}{\pi} \frac{u^2}{2} \bigg|_0^{\sqrt{2}/2} = \frac{2}{\pi} \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{2}{\pi} \cdot \frac{2}{4} = \frac{1}{\pi} \Rightarrow f_{\text{ave}} = \frac{1}{\pi}$$