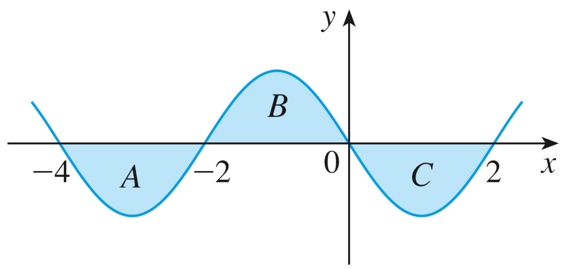
1. The following problems are not related.

(a)(11pts) Suppose the acceleration of an object at any time t is given by $a(t) = 3t^2 - 4t$ m/s^2, $t \geq 0$. Find the velocity, $v(t)$, at any time t if $v(1) = 1$ m/s. Show all work.

(b)(11pts) Each of the regions A, B and C bounded by the graph of $f(x)$ and the x-axis has an area of 5, see below. Find the value of $\int_2^0 \left[\pi^2 f(x) - 4\right]\, dx$.



Problem 1: Each of the regions A, B and C pictured above has an area of 5.

(c)(4pts) Using right endpoints and subintervals of equal width, which of the limits below is equal to $\int_\pi^{2\pi} \sin(x)\, dx$? (**No justification necessary** - *Choose only* <u>*one*</u> *answer, copy down the entire answer.*)

(A) $\lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(\frac{i\pi}{n}\right)$ (B) $\lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(\pi + \frac{i\pi}{n}\right)$ (C) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi \sin(i\pi/n)}{n}$ (D) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi \sin(\pi + i\pi/n)}{n}$

Solution:

(a)(11pts) Recall that $a(t) = v'(t)$ thus

$$v(t) = \int a(t)\, dt = \int (3t^2 - 4t)\, dt = t^3 - 4 \cdot \frac{t^2}{2} + C = t^3 - 2t^2 + C$$

and $v(1) = 1$ implies

$$1 = v(1) = 1^3 - 2 \cdot 1^2 + C \Rightarrow 1 = -1 + C \Rightarrow C = 2 \Rightarrow \boxed{v(t) = t^3 - 2t^2 + 2}$$

(b)(11pts) First note that $\int_0^2 f(x)\, dx = -5$ thus

$$\int_2^0 \left[\pi^2 f(x) - 4\right]\, dx = -\int_0^2 \left[\pi^2 f(x) - 4\right]\, dx = -\pi^2 \int_0^2 f(x)\, dx + \int_0^2 4\, dx = -\pi^2 \cdot (-5) + 4 \cdot (2 - 0) = \boxed{8 + 5\pi^2}$$

(c)(4pts) $\boxed{\text{Choice D.}}$ Note that $\int_a^b f(x)dx = \int_\pi^{2\pi} \sin(x)\, dx$ and $\Delta x = \frac{b-a}{n} = \frac{2\pi - \pi}{n} = \frac{\pi}{n}$ and $x_i = a + i\Delta x = \pi + i(\pi/n)$ thus

$$\int_\pi^{2\pi} \sin(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} \sin(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \sin(\pi + i\pi/n)\, \frac{\pi}{n}$$

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2. The following problems are not related.

(a)(11pts) Suppose we want to approximate a solution to the equation $3\sin(x) = x$ using Newton's Method. What would the formula for x_{n+1} be? (To get full points for this question you must provide the explicit formula for x_{n+1} in terms of x_n, the generic formula for Newton's Method is <u>not</u> sufficient. You do **not** need to approximate the solution.)

(b)(11pts) Write the expression $\int_2^5 f(x)\,dx + \int_{-2}^2 f(t)\,dt - \int_{-2}^{-1} f(x)\,dx$ as a single integral in the form $\int_a^b f(x)\,dx$.

(c)(4pts) Using differentiation, which one of the choices below can be verified to be equivalent to $\int x\cos(x)\,dx$? (**No justification necessary** - *Choose only* <u>*one*</u> *answer, copy down the entire answer.*)

(A) $\frac{x^2}{2}\sin(x)+C$ (B) $\cos(x)+x\sin(x)+C$ (C) $x\sin(x)+C$ (D) $\frac{\cos^2(x)}{2}+C$ (E) $\sin(x)+C$

Solution:

(a)(11pts) Note that $3\sin(x)=x \Rightarrow 3\sin(x)-x=0$, thus if we let $f(x)=3\sin(x)-x$ then

$$\begin{aligned} x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)} &= \boxed{x_n-\frac{3\sin(x_n)-x_n}{3\cos(x_n)-1}} \\ &= \frac{3x_n\cos(x_n)-x_n-(3\sin(x_n)-x_n)}{3\cos(x_n)-1} = \frac{3[x_n\cos(x_n)-\sin(x_n)]}{3\cos(x_n)-1} \end{aligned}$$

(b)(11pts) Note that in a definite integral the variable of integration is a *dummy variable* and can be replaced with any other variable so we can write $\int_{-2}^2 f(t)\,dt = \int_{-2}^2 f(x)\,dx$ thus

$$\int_2^5 f(x)\,dx + \int_{-2}^2 f(t)\,dt - \int_{-2}^{-1} f(x)\,dx = \int_{-1}^{-2} f(x)\,dx + \int_{-2}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx = \boxed{\int_{-1}^{5} f(x)\,dx}$$

(c)(4pts) $\boxed{\text{Choice B.}}$ Note that

$$\frac{d}{dx}[\cos(x)+x\sin(x)+C] = -\sin(x)+\sin(x)+x\cos(x)+0 = x\cos(x) \Rightarrow \int x\cos(x)\,dx = \cos(x)+x\sin(x)+C$$

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3. The following problems are not related. Show all work.

(a)(12pts) Use the Fundamental Theorem of Calculus to evaluate the definite integral $\int_{-\pi}^{\pi} |3\sin(x)|\,dx$.

(b)(12pts) If $f(x)=\int_4^{x^2} \frac{t-1}{t^2+1}\,dt$, use the Fundamental Theorem of Calculus to find $f'(2)$. Simplify your answer.

Solution:

(a)(12pts) First note that

$$\int_{-\pi}^{\pi} |3\sin(x)|\,dx = \int_{-\pi}^{0} -3\sin(x)\,dx + \int_0^{\pi} 3\sin(x)\,dx$$

now, by the Fundamental Theorem of Calculus, we have

$$\begin{aligned} \int_{-\pi}^{0} -3\sin(x)\,dx + \int_0^{\pi} 3\sin(x)\,dx &= -3\cdot -\cos(x)\Big|_{-\pi}^{0} + 3\cdot -\cos(x)\Big|_0^{\pi} \\ &= 3[\cos(0)-\cos(-\pi)] + 3[-\cos(\pi)+\cos(0)] \\ &= 3[1-(-1)]+3[-(-1)+1] = 6+6 = \boxed{12} \end{aligned}$$

(alternately, note that $f(x)=|3\sin(x)|$ is an even function thus $\int_{-\pi}^{\pi} |3\sin(x)|\,dx = 2\int_0^{\pi} 3\sin(x)\,dx = 2\cdot 6 = 12$.)

(b)(12pts) By the Fundamental Theorem of Calculus we have

$$f'(x) = \frac{d}{dx}\left[\int_4^{x^2} \frac{t-1}{t^2+1}\,dt\right] = \frac{x^2-1}{(x^2)^2+1}\cdot 2x = \frac{2x(x^2-1)}{x^4+1} \Rightarrow f'(2) = \frac{2\cdot 2\cdot (2^2-1)}{2^4+1} = \frac{4\cdot 3}{17} = \boxed{\frac{12}{17}}$$

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4. The following problems are not related. Show all work.

(a)(12pts) Use u-substitution to evaluate the definite integral $\displaystyle\int_0^4 \frac{\sqrt[3]{1+\sqrt{x}}}{\sqrt{x}}\,dx$. Show all work

(b)(12pts) Find the *average value* of $f(x) = \dfrac{\sin(x)}{\sec(x)}$ on the interval $[0, \pi/4]$. Justify your answer.

Solution:

(a)(12pts) If we let $u = 1 + \sqrt{x} \Rightarrow du = \dfrac{1}{2\sqrt{x}}dx \Rightarrow 2du = \dfrac{1}{\sqrt{x}}dx$ also note that $x = 0 \Rightarrow u = 1 + \sqrt{0} = 1$ and $x = 4 \Rightarrow u = 1 + \sqrt{4} = 3$ thus

$$\int_0^4 \frac{\sqrt[3]{1+\sqrt{x}}}{\sqrt{x}}\,dx = \int_0^4 \left(1+\sqrt{x}\right)^{1/3} \cdot \frac{1}{\sqrt{x}}\,dx = \int_1^3 u^{1/3} \cdot 2du = 2 \cdot \frac{3}{4}u^{4/3}\Big|_1^3 = \boxed{\frac{3}{2}\left(3^{4/3} - 1\right)}$$

(b)(12pts) Recall that the average value of $f(x)$ over the interval $[a, b]$ is given by

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{\pi/4 - 0}\int_0^{\pi/4} \frac{\sin(x)}{\sec(x)}\,dx = \frac{4}{\pi}\int_0^{\pi/4} \sin(x)\cos(x)\,dx$$

now if we let $u = \sin(x) \Rightarrow du = \cos(x)dx$ and $x = 0 \Rightarrow u = \sin(0) = 0$ and $x = \pi/4 \Rightarrow u = \sin(\pi/4) = \sqrt{2}/2$ thus

$$\frac{4}{\pi}\int_0^{\pi/4} \sin(x)\cos(x)\,dx = \frac{4}{\pi}\int_0^{\sqrt{2}/2} u\,du = \frac{4}{\pi}\cdot\frac{u^2}{2}\Big|_0^{\sqrt{2}/2} = \frac{2}{\pi}\left[\left(\frac{\sqrt{2}}{2}\right)^2 - 0\right] = \frac{2}{\pi}\cdot\frac{2}{4} = \frac{1}{\pi} \Rightarrow \boxed{f_{ave} = 1/\pi}$$
