1. The following problems are not related.

(a)(12pts) Find the linearization $L(x)$ of the function $f(x) = \sqrt{1-x}$ at $a = 0$ and use it to approximate $\sqrt{0.9}$.

(b)(12pts) Find dy/dx by implicit differentiation given that $y\cos(x) = x^2 + y^2$. Simplify your answer.

(c)(4pts) Which choice below is the correct derivative of $g(x) = \sin^2\left(\frac{x}{\pi}\right)$? (**No justification necessary** - *Choose only* <u>*one*</u> *answer, copy down the entire answer.*)

(A) $g'(x) = \frac{2}{\pi}\sin\left(\frac{x}{\pi}\right)$ (B) $g'(x) = \frac{2}{\pi}\cos\left(\frac{x}{\pi}\right)$ (C) $g'(x) = \frac{2}{\pi}\sin\left(\frac{x}{\pi}\right)\cos\left(\frac{x}{\pi}\right)$ (D) $g'(x) = 2\sin\left(\frac{x}{\pi}\right)\cos\left(\frac{x}{\pi}\right)\left(\frac{\pi - x}{\pi^2}\right)$

Solution: (a)(12pts) Here $f'(x) = -\frac{1}{2}(1-x)^{-1/2}$ and so

$$L(x) = f(0) + f'(0)(x-0) = 1 - \frac{x}{2} \Rightarrow \boxed{L(x) = 1 - \frac{x}{2}}$$

and so $\boxed{\sqrt{0.9} = f(0.1) \approx L(0.1) = 1 - \frac{0.1}{2} = 0.95 = 19/20.}$

(b)(12pts) Differentiating both sides of the equation $y\cos(x) = x^2 + y^2$ yields

$$y'\cos(x) - y\sin(x) = 2x + 2yy' \Rightarrow \big(\cos(x) - 2y\big)y' = 2x + y\sin(x) \Rightarrow \boxed{y' = \frac{2x + y\sin(x)}{\cos(x) - 2y}}$$

(c)(4pts) $\boxed{\text{Choice C}}$ According the Chain Rule,

$$\left[\sin^2\left(\frac{x}{\pi}\right)\right]' = 2\sin\left(\frac{x}{\pi}\right)\cos\left(\frac{x}{\pi}\right)\cdot\frac{1}{\pi} \quad \text{thus} \quad \boxed{g'(x) = \frac{2}{\pi}\sin\left(\frac{x}{\pi}\right)\cos\left(\frac{x}{\pi}\right)}$$

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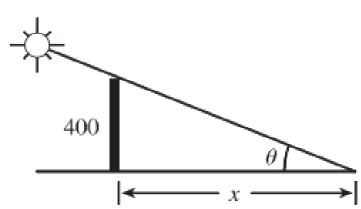
2. The following problems are not related. Justify your answers, show all work.

(a)(12pts) In the diagram below, the angle of elevation of the sun, θ, is decreasing at a rate of 0.25 rad/h. How fast is the shadow cast by a 400-ft-tall building changing when the angle of elevation of the sun is $\pi/6$? Include units in your answer.



Problem 2: The variable x above represents the length of the building's shadow.

(b)(*i*)(6pts) State the Mean Value Theorem. (*ii*)(6pts) Find all numbers c that satisfy the conclusion of the Mean Value Theorem for the function $f(x) = 1/x$ over the interval $[1, 3]$.

Solution: (a)(12pts) We are given $d\theta/dt = -0.25$ and from the diagram we have

$$\cot(\theta) = \frac{x}{400} \Rightarrow x = 400\cot(\theta) \Rightarrow \frac{dx}{dt} = -400\csc^2(\theta)\frac{d\theta}{dt}\bigg|_{\theta=\pi/6} = -400\cdot(2)^2\cdot(-0.25) = \boxed{400\text{ft/h}}$$

(b)(i)(6pts) Mean Value Theorem: If $\boxed{f(x) \text{ is continuous on } [a,b]}$ and $\boxed{\text{differentiable on } (a,b)}$ then

$\boxed{\text{there exists a number } c \text{ in the interval } (a,b)}$ such that $\boxed{f'(c) = \dfrac{f(b)-f(a)}{b-a}.}$

(b)(ii)(6pts) Note that

$$f'(x) = -\frac{1}{x^2} \quad \text{and} \quad \frac{f(3)-f(1)}{3-1} = \frac{1/3-1}{2} = \frac{-2/3}{2} = -\frac{1}{3} \Rightarrow -\frac{1}{x^2} = -\frac{1}{3} \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$$

and so $\boxed{x = \sqrt{3} \text{ satisfies the Mean Value Theorem for } f(x) = 1/x \text{ over the interval } [1,3].}$

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3. The following problems are not related.

(a)(12pts) Suppose x represents the edge length of a metal cube. (i)(6pts) If $V(x) = x^3$ find dV, the differential of V. (ii)(6pts) Suppose the edge of the cube was originally found to be 10 cm in length but expands due to heat to 10.01 cm, use differentials to estimate the change in the volume, ΔV.

(b)(12pts) Find the absolute minimum and maximum values of $f(x) = x^3 + 6x^2 + 1$ on the interval $[-1,1]$. Give your answer in the form (x,y). Show all work, justify your answers and clearly label your answers.

(c)(4pts) Which choice below is the correct derivative of $f(x) = \dfrac{3x+2}{2x+3}$? (**No justification necessary** - *Choose only* *<u>one</u> answer, copy down the entire answer.*)

(A) $f'(x) = -\dfrac{1}{(2x+3)^2}$ (B) $f'(x) = \dfrac{3}{2}$ (C) $f'(x) = \dfrac{13}{(2x+3)^2}$ (D) $f'(x) = -\dfrac{5}{(2x+3)^2}$ (E) $f'(x) = \dfrac{5}{(2x+3)^2}$

Solution:

(a)(i)(6pts) If $V = x^3$ then $\boxed{dV = 3x^2 dx.}$

(a)(ii)(6pts) Note that $\Delta x = 10.01 - 10 = 0.01$ thus

$$\Delta V \approx dV = 3x^2 dx\Big|_{x=10, dx=\Delta x} = 3(10)^2(0.01) = 3 \text{ cm}^3$$

so the $\boxed{\text{volume will increase by approximately 3 cm}^3.}$

(b)(12pts) Here we have

$$f(x) = x^3 + 6x^2 + 1 \Rightarrow f'(x) = 3x^2 + 12x = 3x(x+4) \quad \text{and so} \quad f'(x) = 0 \Rightarrow x = 0, -4$$

but $x = -4$ is not in the interval $[-1,1]$. Now plugging in the endpoints and critical points into $f(x)$ yields

$$f(-1) = -1 + 6 + 1 = 6,\ f(0) = 1,\ \text{ and } f(1) = 1 + 6 + 1 = 8$$

thus an $\boxed{\text{absolute maximum occurs at } (1,8) \text{ and and absolute minimum occurs at } (0,1).}$

(c)(4pts) $\boxed{\text{Choice E.}}$ Using the Quotient Rule, we have

$$\left[\frac{3x+2}{2x+3}\right]' = \frac{3\cdot(2x+3) - 2\cdot(3x+2)}{(2x+3)^2} = \frac{(6x+9)-(6x+4)}{(2x+3)^2} = \frac{5}{(2x+3)^2} \quad \text{thus} \quad \boxed{f'(x) = \frac{5}{(2x+3)^2}}$$

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4. The following problems are not related. Justify your answers, show all work.

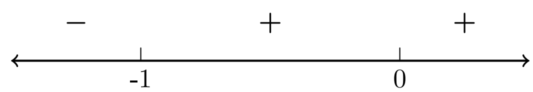
(a)(12pts) Given $g(x) = x^{1/3}(x+4)$, $g'(x) = \frac{4}{3}x^{-2/3}(x+1)$ and $g''(x) = \frac{4}{9}x^{-5/3}(x-2)$. (i)(6pts) Give local maximum and minimum values of $g(x)$ as an ordered pair (x, y) and justify your answer with either the 1st or 2nd Derivative Test. Clearly label your answers. (ii)(6pts) Give any inflection points of $g(x)$ as an ordered pair (x, y) and justify your answer.

(b)(8pts) In your blue book clearly sketch the graph of a function $f(x)$ that satisfies all the following properties (label all extrema, inflection points and asymptotes if any):

- $f(x)$ has a vertical asymptote at $x = 0$ and $f(-2) = -1$
- $f'(x) > 0$ if $x < -2$ and $f'(x) < 0$ if $x > -2$ (but $x \neq 0$)
- $f''(x) < 0$ if $x < 0$ and $f''(x) > 0$ if $x > 0$

Solution: (a)(i)(6pts) First note that the domain of $g(x)$ is all real numbers. Now $g'(x) = 0$ implies $x = -1$ and $g'(x)$ is undefined when $x = 0$, and using the 1st Derivative Test for $g'(x) = \frac{4}{3}x^{-2/3}(x+1)$ yields

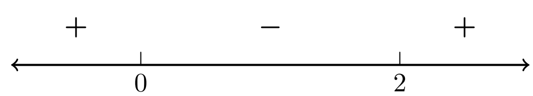
Sign chart for $g'(x)$



thus we have a $\boxed{\text{local minimum at } (-1, g(-1)) = (-1, -3) \text{ and there are no local maxima.}}$ Note that we could use the 2nd Derivative Test to classify $x = -1$ as a local minimum since $g''(-1) = 4/3 > 0$ but we cannot use the 2nd Derivative Test to classify $x = 0$ since $g''(x)$ is not defined there.

(a)(ii)(6pts) If $g''(x) = 0$ then $x = 2$ and $g''(x)$ is undefined when $x = 0$, now using a sign chart for $g''(x)) = \frac{4}{9}x^{-5/3}(x-2)$ yields

Sign chart for $g''(x)$



so we have $\boxed{\text{inflection points at } (0, g(0)) = (0, 0) \text{ and } (2, g(2)) = (2, 6\sqrt[3]{2}).}$

(b)(8pts) The graph must have a local max at the point $(-2, -1)$ and must be increasing if $x < -2$ and decreasing otherwise and $f(x)$ must be concave down if $x < 0$ and concave up otherwise and must have a vertical asymptote at $x = 0$. The graph of $f(x)$ could look like, for example: