1. The following problems are not related.

(a)(10pts) Suppose $f(x) = \sqrt{x}$ and $g(x) = \sqrt{4 - x^2}$. Find $(g \circ f)(x)$ and express the domain of this function in interval notation.

(b)(10pts) Suppose we know that the function $p(x)$ is an *even* function. Show that the function $q(x) = x^3 + \sin(x) + xp(x)$ is an *odd* function. Justify your answer.

(c)(5pts) Which choice below would result in shifting the graph of $y = s(t)$ one unit to the right and then reflecting it about the y-axis? (**No justification necessary** - *Choose only* <u>*one*</u> *answer, copy down the entire answer.*)

(A) $y = -s(t) - 1$ (B) $y = s(-(t+1))$ (C) $y = s(-(t-1))$ (D) $y = -s(t+1)$ (E) $y = s(-t) - 1$

Solution: (a)(10pts) Proceeding by definition, we have that

$$(g \circ f)(x) = g(f(x)) = \underbrace{g(\sqrt{x})}_{\text{need } x \geq 0} = \sqrt{4 - (\sqrt{x})^2} = \underbrace{\sqrt{4 - x}}_{\text{need } x \leq 4} \text{ with doman } [0, 4].$$

(b)(10pts) Note that since $p(x)$ is even we have that $p(-x) = p(x)$ and recall that $\sin(x)$ is an odd function thus

$$q(-x) = (-x)^3 + \sin(-x) + (-x)p(-x) = -x^3 - \sin(x) - xp(x) = -(x^3 + \sin(x) + xp(x)) = -q(x)$$

thus $q(-x) = -q(x)$ and so $q(x)$ is an odd function.

(c)(5pts) Choice B. Note that shifting the graph of $y = s(t)$ one unit to the right and then reflecting the graph about the y-axis is the same as reflecting the graph of $s(t)$ about the y-axis, *i.e.* $s(-t)$, and then shifting the graph one unit to the left, thus we have $s(-(t+1))$. (Note Choice C is incorrect since the transformation $y = s(-(t-1))$ shifts the graph on unit to the right and then reflects the graph about the vertical line $t = 1$ not the y-axis.)

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2. The following problems are not related.

(a)(10pts) Use the Squeeze Theorem to evaluate the following limit: $\lim_{x \to 1} (x-1)^2 \sin\left(\frac{1}{x-1}\right)$. Show all work, explain your answer.

(b)(12pts) Suppose that $f(x) = \begin{cases} \dfrac{4x+1}{2-x}, & \text{if } x \leq 0 \\ x + \dfrac{1}{x}, & \text{if } x > 0 \end{cases}$, use limits to find all horizontal and vertical asymptotes of $f(x)$. Show all work.

Solution: (a)(10pts) Note that

$$-1 \leq \sin\left(\frac{1}{x-1}\right) \leq 1 \Longrightarrow -(x-1)^2 \leq (x-1)^2 \sin\left(\frac{1}{x-1}\right) \leq (x-1)^2$$

and since $\lim_{x \to 1} -(x-1)^2 = \lim_{x \to 1} (x-1)^2 = 0$ thus, by the <u>Squeeze Theorem</u>, $\lim_{x \to 1} (x-1)^2 \sin\left(\frac{1}{x-1}\right) = 0$.

(b)(12pts) For the horizontal asymptotes, note that

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} x + \frac{1}{x} = +\infty \Rightarrow \text{ no asymptote in this direction}$$

and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{4x+1}{-x+2} = \lim_{x\to-\infty} \frac{4x(1+1/4x)}{-x(1-2/x)} = \lim_{x\to-\infty} \frac{4\cancel{x}(1+1/4x)}{-\cancel{x}(1+2/x)} = \frac{4}{-1} = -4 \Rightarrow \boxed{\text{H.A. at } y=-4}$$

and for the vertical asymptote, note that we just need to check the limit as $x \to 0^+$, we have

$$\lim_{x\to0^+} f(x) = \lim_{x\to0^+} x + \frac{1}{x} = +\infty \Rightarrow \boxed{\text{V.A. at } x=0}$$

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3. The following problems are not related.

(a)(12pts) Find the real number a so that the function $f(x) = \begin{cases} \dfrac{3\sin(1+x)}{1+x}, & \text{if } x \neq -1 \\ ax+8, & \text{if } x=-1 \end{cases}$ is continuous for all real numbers. Be sure to show that all three conditions of continuity have been satisfied.

(b)(12pts) The function $g(x) = \dfrac{x+10}{|x|+2}$ has two horizontal asymptotes. They are $y=1$ and $y=-1$. Use a theorem from class to show that $g(x)$ crosses one of it's horizontal asymptotes on the interval $[-10,0]$. Clearly explain your answer.

(c)(5pts) For which one of the 4 choices below is the following true: $\lim_{x\to3} f(x) = -2$. (**No justification necessary** - *Choose only <u>one</u> answer, copy down the entire answer.*)

(A) $f(x) = \dfrac{-6\sin(\pi x/6)}{x}$ (B) $f(x) = \dfrac{-2(x-3)}{|x-3|}$ (C) $f(x) = \dfrac{-2x^2-3x+4}{x^2-1}$ (D) $f(x) = \begin{cases} \cos(\pi x/6)-2, & \text{if } x \leq 3 \\ 2, & \text{if } x > 3 \end{cases}$

Solution: (a)(12pts) By observation note that $f(x)$ is continuous for all $x \neq -1$. Now, note that at $x=-1$ we have

(1) $f(-1) = -a+8$

(2) $\lim_{x\to-1} f(x) = \lim_{x\to-1} \dfrac{3\sin(1+x)}{1+x} = 3\left(\lim_{x\to-1} \dfrac{\sin(1+x)}{1+x}\right) = 3(1) = 3$

(3) and, finally, we need $\lim_{x\to-1} f(x) = f(-1)$ which implies $-a+8=3 \Rightarrow a=5$

thus $\boxed{\text{we need } a=5 \text{ for } f(x) \text{ to be continuous for all real numbers.}}$

(b)(12pts) Note that $g(-10)=0$ and $g(0)=5$ and since $g(x)$ is a <u>continuous</u> function for all x and since $g(-10) \leq 1 \leq g(0)$, by the <u>Intermediate Value Theorem</u>, there exists some number c in $(-10,0)$ such that $g(c)=1$ thus we see that $g(x)$ crosses one of its horizontal asymptotes.

(c)(5pts) $\boxed{\text{Choice A.}}$ Direct calculation of the limit shows $\lim_{x\to3} f(x) = \lim_{x\to3} \dfrac{-6\sin(\pi x/6)}{x} = \dfrac{-6\sin(\pi/2)}{3} = -\dfrac{6}{3} = -2$.

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4. The following problems are not related.

(a)(12pts) For this problem, use the <u>*limit definition of the derivative*</u> to find the derivative of $f(x) = \dfrac{1}{\sqrt{x}}$, show all work.

(b) Consider the function $g(x) = \sqrt[3]{x^2-1}$ with derivative $g'(x) = \dfrac{2x}{3(x^2-1)^{2/3}}$.

(i)(6pts) Find the equation of the tangent line to $g(x) = \sqrt[3]{x^2-1}$ at the point $x=3$. Simplify your answer.

(ii)(6pts) For what values of x is the function $g(x) = \sqrt[3]{x^2-1}$ *differentiable*? Justify your answer.

Solution: (a)(12pts) Note that

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{1/\sqrt{x+h} - 1/\sqrt{x}}{h} = \lim_{h\to 0} \frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}}$$

now multiplying by the conjugate yields

$$\begin{aligned}
\lim_{h\to 0} \frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} \cdot \frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}} &= \lim_{h\to 0} \frac{x-(x+h)}{h\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})} \\
&= \lim_{h\to 0} \frac{-\not{h}}{\not{h}\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})} = \frac{-1}{\sqrt{x}\cdot\sqrt{x}(\sqrt{x}+\sqrt{x})} = \frac{-1}{x\cdot(2\sqrt{x})} = -\frac{1}{2x^{3/2}}
\end{aligned}$$

thus $\boxed{f'(x) = -1/2x^{3/2}.}$

(b)(i)(6pts) The equation of the tangent line is

$$y = g(3) + g'(3)(x-3) = \sqrt[3]{8} + \frac{6}{3\cdot 8^{2/3}}(x-3) = 2 + \frac{6}{12}(x-3) = \frac{x}{2} + \frac{1}{2} \Rightarrow \boxed{y = \frac{x}{2} + \frac{1}{2}}$$

(b)(ii)(6pts) First note that $g(x)$ is continuous for all values of x. Now, since $g'(x)$ is undefined at $x=-1$ and $x=1$ we see that $\boxed{g(x) \text{ is differentiable for all } x \neq -1 \text{ and } x \neq 1}$ (and since $\lim_{x\to -1} g'(x) = +\infty$ and $\lim_{x\to 1} g'(x) = +\infty$ there are vertical tangents at $x=-1$ and $x=1$)