1. (a)(13pts) Use the Squeeze Theorem to evaluate the limit: $\lim_{x\to 0} 2|x|\sin(1/x^2)$. Show all work.

   (b)(13pts) Evaluate the limit $\lim_{x\to\infty}(e^x+x)^{1/x}$. Show all work, justify your answer.

   (c)(6pts) For this problem choose only one answer from below, **no justification necessary**- *Copy down the entire answer*. Suppose $g(x) = \dfrac{(x-2)\ln(x)}{x^2+x-6}$, then $g(x)$ has a *vertical asymptote* at:

   (A) $x=-3, x=2$ (B) $x=-3$ (C) $x=-3, x=0$ (D) $x=0$ (E) $x=0, x=2$

   Solution:

   (a)(13pts) Note that

   $$-1 \le \sin(1/x^2) \le 1 \Rightarrow -2|x| \le \sin(1/x^2) \le 2|x|$$

   and $\lim_{x\to 0} -2|x| = \lim_{x\to 0} 2|x| = 0$ thus $\boxed{\lim_{x\to 0} 2|x|\sin(1/x^2) = 0}$ by the Squeeze Theorem.

   (b)(13pts) First note that this limit is of the indeterminate type "∞^0". Letting $y = \lim_{x\to\infty}(e^x+x)^{1/x}$, we have

   $$\ln(y) = \ln[\lim_{x\to\infty}(e^x+x)^{1/x}] = \lim_{x\to\infty}\ln[(e^x+x)^{1/x}] = \lim_{x\to\infty}\frac{\ln(e^x+x)}{x}$$
   $$\stackrel{L'H}{=} \lim_{x\to\infty}\frac{e^x+1}{e^x+x} \stackrel{L'H}{=} \lim_{x\to\infty}\frac{e^x}{e^x+1} \stackrel{L'H}{=} \lim_{x\to\infty}\frac{\cancel{e^x}}{\cancel{e^x}} = 1$$

   thus $\ln(y)=1$ and so $y=e$, *i.e.* $\boxed{\lim_{x\to\infty}(e^x+x)^{1/x} = e.}$

   (c)(6pts) $\boxed{\text{Choice D.}}$ Note that

   $$g(x) = \frac{(x-2)\ln(x)}{x^2+x-6} = \frac{(x-2)\ln(x)}{(x-2)(x+3)} = \frac{\ln(x)}{x+3}$$

   and so the domain of $g(x)$ must satisfy $x>0$, $x\neq 2$ and $x\neq -3$ and clearly we have a removable discontinuity at $x=2$ thus the only vertical asymptote of $g(x)$ is $\boxed{x=0}$.

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2. (a)(13pts) Consider the function $f(x) = \begin{cases} \sinh\left(\pi e^x - \cos^{-1}(x-1)\right), & \text{if } x \le 0 \\ x\ln(x), & \text{if } x > 0 \end{cases}$.

   (*i*) Evaluate the limit $\lim_{x\to 0^+} f(x)$. Show all work.

   (*ii*) Is the function $f(x)$ continuous at $x=0$? Justify your answer with a limit. (If $f(x)$ is not continuous at $x=0$ then what type of discontinuity is present at $x=0$? Explain.)

   (b)(13pts) Julien is jogging around a circular track of radius 50m. In a coordinate system with origin at the center of the track, Julien's x-coordinate is changing at a rate of -1.25 m/s when their coordinate is $(40,30)$. Find dy/dt at this moment. (Recall that the equation of a circle with radius r is $x^2+y^2=r^2$.)

   (c)(6pts) In your bluebook, clearly sketch the graph of $g(x) = 3x^{1/3} - x = x^{1/3}(3-x^{2/3})$, note that $g'(x) = \dfrac{1}{x^{2/3}} - 1$ and $g''(x) = -\dfrac{2}{3x^{5/3}}$. Label all <u>intercepts</u>, <u>extrema</u>, <u>inflection points</u> and <u>asymptotes</u>. (Recall that $\sqrt{3}\approx 1.7$)

   Solution:

   (a)(*i*) Note that this limit yields the indeterminate form "$0\cdot -\infty$" and so

   $$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+}\frac{\ln(x)}{1/x} \stackrel{L'H}{=} \lim_{x\to 0^+}\frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -\frac{x^2}{x} = \lim_{x\to 0^+} -x = 0$$

(a)(*ii*) Yes, $f(x)$ is continuous at $x=0$. Note that

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \sinh\left(\pi e^x - \cos^{-1}(x-1)\right) = \sinh(\underbrace{\pi\cdot e^0}_{\pi} - \underbrace{\cos^{-1}(-1)}_{\pi}) = \underbrace{\sinh(0)}_{f(0)} = 0 = \lim_{x\to 0^+} f(x)$$

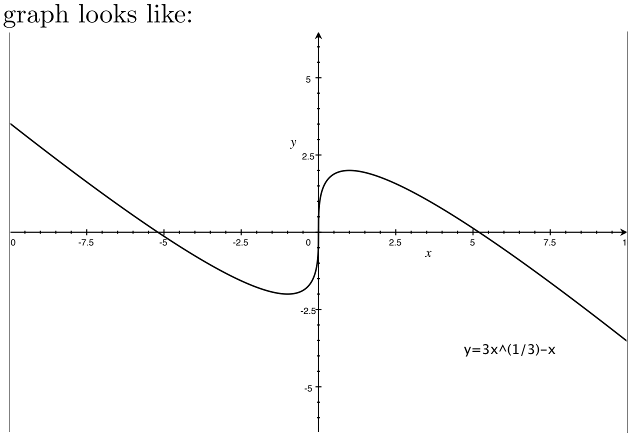
thus we see that $\boxed{\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) \text{ so } f(x) \text{ is continuous at } x=0.}$

(b)(13pts) We wish to find dy/dt when $(x,y)=(40,30)$ given that $dx/dt=-1.25$ m/s. Now note

$$x^2+y^2=(50)^2 \Rightarrow 2x\frac{dx}{dt}+2y\frac{dy}{dt}=0 \Rightarrow \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = -\frac{40}{30}\cdot(-1.25) = 5/3 \text{ m/s}$$

So $\boxed{\text{Julien's } y\text{-coordinate is increasing at the rate of } 5/3 \text{ m/s}}$ at the point $(40,30)$.

(c)(6pts) Note that we have intercepts $(0,0)$, $(0,\pm 3\sqrt{3})$, local min at $(-1,-2)$ and a local max at $(1,2)$, there is an inflection point at $(0,0)$, the graph looks like:



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3. (a)(13pts) Evaluate the integral: $\displaystyle\int_0^{\cosh^{-1}(\sqrt{3})} \frac{\sinh(x)}{1+(\cosh(x))^2}\,dx$. Show all work.

(b)(13pts) Find the area between the positive function $f(x)=e^x\sqrt{e^x+1}$ and the x-axis from $x=\ln(3)$ to $x=\ln(8)$.

(c)(7pts) Which of the five choices <u>given below</u> is the limit $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\sin^{-1}\left(\frac{i}{n}\right)\frac{i}{n^2}$ equivalent to? Choose only one answer, **no justification necessary** - *be sure to copy down the <u>entire answer</u>.*

(A) $\displaystyle\int_0^1 \sin^{-1}(x)\,dx$ (B) $\displaystyle\int_0^1 \frac{x}{\sin(x)}\,dx$ (C) $\displaystyle\int_0^1 x\sin(x/n)\,dx$ (D) $\displaystyle\int_0^1 x\sin^{-1}(x)\,dx$ (E) $\displaystyle\int_{-1}^0 x\sin^{-1}(x)\,dx$

Solution:

(a)(13pts) Using the u-substitution $u=\cosh(x) \Rightarrow du=\sinh(x)dx$ and if $x=0$ then $u=\cosh(0)=1$ and if $x=\cosh^{-1}(\sqrt{3})$ then $u=\sqrt{3}$ and so

$$\int_0^{\cosh^{-1}(\sqrt{3})} \frac{\sinh(x)}{1+(\cosh(x))^2}\,dx = \int_1^{\sqrt{3}} \frac{1}{1+u^2}\,du = \tan^{-1}(u)\Big|_1^{\sqrt{3}} = \tan^{-1}(\sqrt{3})-\tan^{-1}(1) = \frac{\pi}{3}-\frac{\pi}{4} = \frac{\pi}{12}$$

(b)(13pts) We wish to evaluate the integral $\displaystyle\int_{\ln(3)}^{\ln(8)} e^x\sqrt{e^x+1}\,dx$. If we let $u=e^x+1 \Rightarrow du=e^x dx$ and so

$$\int_{\ln(3)}^{\ln(8)} e^x\sqrt{e^x+1}\,dx = \int_4^9 \sqrt{u}\,du = \frac{2}{3}u^{3/2}\Big|_4^9 = \frac{2}{3}\left(9^{3/2}-4^{3/2}\right) = \frac{2}{3}(27-8) = \frac{38}{3}$$

(c)(7pts) $\boxed{\text{Choice D.}}$ Note that for choice D we have $\Delta x = (b-a)/n = 1/n$ and $x_i = a+i\Delta x = i/n$ and so using a Rieman Sum with right endpoints we see that

$$\boxed{\int_0^1 x\sin^{-1}(x)\,dx} = \lim_{n\to\infty}\sum_{i=1}^{n} x_i\sin^{-1}(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{i}{n}\sin^{-1}\left(\frac{i}{n}\right)\frac{1}{n} = \lim_{n\to\infty}\sum_{i=1}^{n}\sin^{-1}\left(\frac{i}{n}\right)\frac{i}{n^2}$$

4. (a)(13pts) Use logarithmic differentiation to find dy/dx: $(i)\ y = \dfrac{x\sin(x)}{\sqrt{\sec(x)}}$ $(ii)\ y = (\tan^{-1}(x))^x$

(b)(13pts) A bacteria culture initially contains 100 cells and grows at a rate proportional to its size. After an hour the population has increased to 420. (i) Find the relative growth rate of the population. (ii) Find the number of bacteria after 3 hours.

(c)(7pts) Which of the five choices given below is the antiderivative $\int \frac{5^x}{1+5^x}\,dx$ equivalent to? (Choose only one answer, **no justification necessary** - *be sure to copy down the entire answer.*)

(A) $\tan^{-1}(5^{x/2})+C$ (B) $\dfrac{1}{(1+5^x)^2}+C$ (C) $\ln\left(\dfrac{1+5^x}{5}\right)+C$ (D) $\ln(1+5^x)-\ln(5)+C$ (E) $\dfrac{\ln(1+5^x)}{\ln(5)}+C$

Solution:

(a)(6pts)(i) Note that

$$y = \frac{x\sin(x)}{\sqrt{\sec(x)}} \Rightarrow \ln(y) = \ln\left(\frac{x\sin(x)}{\sqrt{\sec(x)}}\right) = \ln(x) + \ln(\sin(x)) - \frac{1}{2}\ln(\sec(x))$$

and so differentiating on both sides yields

$$\frac{y'}{y} = \frac{1}{x} + \frac{\cos(x)}{\sin(x)} - \frac{1}{2}\cdot\frac{\sec(x)\tan(x)}{\sec(x)} \Rightarrow y' = y\left(\frac{1}{x} + \cot(x) - \frac{\tan(x)}{2}\right) \Rightarrow \boxed{y' = \frac{x\sin(x)}{\sqrt{\sec(x)}}\left(\frac{1}{x} + \cot(x) - \frac{\tan(x)}{2}\right)}$$

(a)(7pts)(ii) Here we have

$$y = (\tan^{-1}(x))^x \Rightarrow \ln(y) = \ln[(\tan^{-1}(x))^x] = x\ln(\tan^{-1}(x))$$

and so differentiating yields

$$\frac{y'}{y} = \ln(\tan^{-1}(x)) + x\cdot\frac{1/1+x^2}{\tan^{-1}(x)} = \ln(\tan^{-1}(x)) + \frac{x}{(1+x^2)\tan^{-1}(x)}$$

and so

$$y' = y\left(\ln(\tan^{-1}(x)) + \frac{x}{(1+x^2)\tan^{-1}(x)}\right) \Rightarrow \boxed{y' = (\tan^{-1}(x))^x\left(\ln(\tan^{-1}(x)) + \frac{x}{(1+x^2)\tan^{-1}(x)}\right)}$$

(b)(9pts)(i) Note that $y = y_0e^{kt}$ where $y_0 = 100$ and so

$$420 = y(1) = 100e^k \Rightarrow e^k = \frac{420}{100} \Rightarrow k = \ln\left(\frac{42}{10}\right) \Rightarrow \boxed{\text{Growth rate is } k = \ln(4.2)}$$

(b)(4pts)(ii) Note that after 3 hours the population is $\boxed{y(3) = 100e^{3\ln(4.2)} = 100(4.2)^3.}$

(c)(7pts) $\boxed{\text{Choice E.}}$ If we let $u = 1+5^x \Rightarrow du = 5^x\ln(5)\,dx \Rightarrow du/\ln(5) = 5^x dx$ and so

$$\int\frac{5^x}{1+5^x}\,dx = \frac{1}{\ln(5)}\int\frac{1}{u}\,du = \frac{1}{\ln(5)}\cdot\ln|u| + C = \boxed{\frac{\ln(1+5^x)}{\ln(5)}+C}$$

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5. (20pts) Answer either **ALWAYS TRUE** or **FALSE**. You do NOT need to justify your answer. (*Don't just write down "A.T." or "F", completely write out the words "ALWAYS TRUE" or "FALSE" depending on your answer.*)

(a)(5 pts) Suppose $f(x)$ is continuous for all x, then $f(x)$ is differentiable at $x=a$ if $\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}$ exists.

(b)(5 pts) The linearization of $g(x) = \ln(x+e)$ centered at $a = e$ is $L(x) = \ln(2) + \dfrac{x}{2e} - \dfrac{1}{2}$.

(c)(5 pts) The Mean Value Theorem applies to $h(x) = -1/x$ on $[-3, -1/2]$ and the value of c that satisfies the Mean Value Theorem is $c = -2/3$.

(d)(5 pts) If $f(x) = \int_2^x \sqrt{1+3^t}dt$, then $f(2) = 0$ so $f^{-1}(0) = 2$ and $(f^{-1})'(0) = \dfrac{1}{\sqrt{10}}$.

Solution: (a) A.T. (b) F (c) F (d) A.T.

Discussion:

(a) If $f(x)$ is continuous at $x = a$, then, by definition, $f(x)$ is differentiable at $x = a$ if and only if $\lim_{x\to a} \dfrac{f(x) - f(a)}{x - a}$ exists $\Rightarrow$ A.T.

(b) Note that

$$L(x) = g(e) + g'(e)(x - e) = \ln(2e) + \frac{1}{2e}(x - e) = \ln(2e) + \frac{x}{2e} - \frac{1}{2} \neq \ln(2) + \frac{x}{2e} - \frac{1}{2} \Rightarrow \text{ F.}$$

(c) Here we have the interval $(a, b) = (-3, -1/2)$ and so

$$\frac{h(b) - h(a)}{b - a} = \frac{h(-1/2) - h(-3)}{-1/2 - (-3)} = \frac{2 - 1/3}{3 - 1/2} = \frac{5/3}{5/2} = \frac{2}{3} = \frac{1}{c^2} = h'(c)$$

and so $c = \pm\sqrt{3/2} \neq -2/3 \Rightarrow$ F.

(d) Note that $f'(x) = \sqrt{1 + 3^x}$ and so

$$(f^{-1})'(0) = \frac{1}{f'(f^{-1}(0))} = \frac{1}{f'(2)} = \frac{1}{\sqrt{1+3^2}} = \frac{1}{\sqrt{10}} \Rightarrow \text{ A.T.}$$