1. (a)(12pts) Suppose we are trying to find the dimensions of a rectangle with area 1000 m^2 whose perimeter is to be as small as possible using optimization. Suppose x and y represent the dimensions of the rectangle. Answer the following questions:

   (i)(3pts) Is this is a *minimization* or *maximization* problem? Write down a function in terms of x and y that you would minimize (or maximize). Your answer should be in terms of the variables x and y only.

   (ii)(3pts) Write down an equation that relates the variables x and y.

   (iii)(6pts) Now using optimization find the value of x and y that satisfy this problem. Justify your answer by classifying your critical point(s) using either the 1st or 2nd Derivative Test.

   (b)(12pts) Suppose we want to approximate the intersection points of $f(x) = x^2$ and $g(x) = \cos(x)$. If we want to use Newton's Method to do this then what would the formula for x_{n+1} be? (To get full points for this question you must provide the explicit formula for x_{n+1} in terms of x_n, the generic formula for Newton's Method is <u>not</u> sufficient. You do **not** need to approximate the point of intersection.)

   Solution:

   (a)(i) We wish to <u>*minimize*</u> the perimeter function $P(x, y) = 2x + 2y$.

   (a)(ii) Since $xy = 1000$ then $y = \dfrac{1000}{x}$.

   (a)(iii) Note that using the equation from part (a)(ii) above we have that

$$P(x) = 2x + \frac{2000}{x} \Longrightarrow P'(x) = 2 - \frac{2000}{x^2} = \frac{2x^2 - 2000}{x^2} = \frac{2(x^2 - 1000)}{x^2}$$

   and $P'(x) = 0$ implies $x = \pm\sqrt{1000}$ and so we have the only critical point in the domain is $x = \sqrt{1000}$. Now we classify this critical point using the <u>1st Derivative Test for Absolute Extrema</u>, note that $P'(x) < 0$ if $x < \sqrt{1000}$ and $P'(x) > 0$ if $x > \sqrt{1000}$ and so we have an absolute minimum and the dimensions that minimize the perimeter are $x \times y = \sqrt{1000}\text{ m} \times \sqrt{1000}\text{ m} = \boxed{10\sqrt{10}\text{ m} \times 10\sqrt{10}\text{ m}}$.

   (b)(12 pts) We wish to approximate $x^2 = \cos(x)$ or $x^2 - \cos(x) = 0$, so if we let $f(x) = x^2 - \cos(x)$ then $f'(x) = 2x + \sin(x)$ and so to implement Newton's Method we use the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - \cos(x_n)}{2x_n + \sin(x_n)} = \boxed{\frac{x_n^2 + x_n\sin(x_n) + \cos(x_n)}{2x_n + \sin(x_n)}}.$$

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2. (a)(12pts) Approximate the area of the region bounded by the function $f(x) = 2 + \sin(x)$, $0 \le x \le \pi$ and the x-axis using four approximating rectangles and taking the sample points to be the right endpoints.

   (b)(12pts) Use the formulas

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad\text{and}\quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

to evaluate the limit $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left[\frac{i}{n}+\frac{i^2}{n^2}\right]$. Show all work.

Solution:

(a)(12pts) Note that $\Delta x = \frac{\pi - 0}{4} = \pi/4$ and so $x_i = \frac{i\pi}{4}$ for $i = 0, 1, \ldots, 4$. Now since we are using right endpoints, $x_i^* = x_i$ and thus we approximate

$$\begin{aligned} A &\approx f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \\ &= \frac{\pi}{4}[(2+\sin(\pi/4)) + (2+\sin(\pi/2)) + (2+\sin(3\pi/4)) + (2+\sin(\pi))] \\ &= \frac{\pi}{4}\left[\left(2+\frac{\sqrt{2}}{2}\right) + (2+1) + \left(2+\frac{\sqrt{2}}{2}\right) + (2+0)\right] = \boxed{\frac{(9+\sqrt{2})\pi}{4}} \end{aligned}$$

(b)(12pts) Note that

$$\begin{aligned} \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left[\frac{i}{n}+\frac{i^2}{n^2}\right] &= \lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{n}\sum_{i=1}^{n} i + \frac{1}{n^2}\sum_{i=1}^{n} i^2\right] \\ &= \lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{n}\cdot\frac{n(n+1)}{2} + \frac{1}{n^2}\cdot\frac{n(n+1)(2n+1)}{6}\right] \\ &= \lim_{n\to\infty}\frac{n^2+n}{2n^2} + \frac{2n^3 + \text{l.o.t}}{6n^3} \overset{DOP}{=} \frac{1}{2} + \frac{2}{6} = \boxed{5/6} \end{aligned}$$

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3. (a)(12pts) Use a definite integral to find the net area of the region bounded by the function $g(x) = x\sec(x^2-1)\tan(x^2-1)$ and the x-axis between $x = 1$ and $x = \sqrt{2}$.

(b)(8pts) If oil leaks from a tank at a rate of $r(t)$ gallons per minute at time t, what does $\int_0^{120} r(t)\,dt$ represent? Explain your answer. (Write your answer in complete sentences.)

(c)(6pts) If $F(x) = \int_2^x f(t)\,dt$, where f is the function whose graph is given below, put the following values in order from smallest to largest: $F(0)$, $F(2)$, $F(3)$. (No justification necessary, box your answer.)

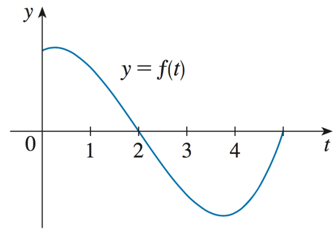
Solution:

(a)(12 pts) We wish to calculate $\int_1^{\sqrt{2}} g(x)dx$ and if we use the u-substitution $u = x^2 - 1$ then $du = 2xdx$, *i.e.* $xdx = du/2$ and so

$$\int_1^{\sqrt{2}} x\sec(x^2-1)\tan(x^2-1)dx = \frac{1}{2}\int_0^1 \sec(u)\tan(u)du = \frac{1}{2}\sec(u)\Big|_0^1 = \boxed{\frac{\sec(1)-1}{2}}$$

(b)(8pts) If oil leaks from a tank at a rate of $r(t)$ gallons per minute then $\int r(t)dt$ is the amount of gallons of oil leaking from the tank and, in particular, $\int_0^{120} r(t)dt$ is the number of gallons of oil that leaked from the tank in the first 120 minutes, *i.e.* in the first two hours.

(c)(6 pts) From the graph we have $\boxed{F(0) < F(3) < F(2)}$



Graph for Problem 3(c).

4. (a)(10pts) Evaluate the indefinite integral $\int x\sqrt{x-1}dx$. Show all work.

(b)(10pts) Find the derivative of the function $f(x) = \int_{1/x}^{2} \sin^2(t)\,dt$. Justify your answer.

(c)(6pts) Suppose the average value of $h(x)$ on the interval $[1, 4]$ is $h_{\text{ave}} = \pi$, which of the following choices below is true? (**No justification necessary** - *Copy down the entire answer*)

(A) $\int_5^{20} h(x/5)dx = 3\pi/5$ (B) $\int_5^{20} h(x/5)dx = 2\pi$ (C) $\int_5^{20} h(x/5)dx = 15\pi$ (D) $\int_5^{20} h(x/5)dx = 8\pi$

Solution:

(a)(10pts) If we use the u-substitution $u = x - 1$ then $du = dx$ and $x = u + 1$ and so

$$\int x\sqrt{x-1}\,dx = \int (u+1)\sqrt{u}\,du = \int (u^{3/2} + u^{1/2})\,du = \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C = \boxed{\frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C}$$

(b)(10 pts) By the Fundamental Theorem of Calculus (PART I) we have

$$\frac{d}{dx}\left(\int_{1/x}^{2} \sin^2(t)\,dt\right) = \frac{d}{dx}\left(-\int_{2}^{1/x} \sin^2(t)\,dt\right) = -\sin^2(1/x) \cdot -\frac{1}{x^2} = \boxed{\frac{\sin^2(1/x)}{x^2}}$$

(c)(6pts) $\boxed{\text{CHOICE C.}}$ Note that

$$\pi = h_{\text{ave}} = \frac{1}{4-1}\int_1^4 h(x)dx \Rightarrow \int_1^4 h(x)dx = 3\pi$$

and now we determine $\int_5^{20} h(x/5)dx$. If we use the u-substitution $u = x/5$ then $du = dx/5$, *i.e.* $dx = 5du$ and so

$$\int_5^{20} h(x/5)dx = 5\int_1^4 h(u)du = 5 \cdot 3\pi = 15\pi \Rightarrow \boxed{\int_5^{20} h(x/5)dx = 15\pi}$$