1. (a)(10pts) Find y' if $y = \tan(x - \cos(x^2))$.

(b)(10pts) If $xg(x) + 2g'(x) = x^2$ and $g'(0) = \pi$ and $g(0) = -10$, find $g''(0)$.

(c)(5pts) Which of the five choices given below is equivalent to y' if $y = \left[\frac{\sin x}{x+1}\right]^2$? Pick only one answer, **no justification necessary** - *be sure to copy down the entire answer, don't just write down the letter of your choice*:

(A) $\frac{2\sin(x)\cos(x)}{2(x+1)}$ (B) $\frac{\cos(x)(x+1) - \sin(x)}{(x+1)^2}$ (C) $\frac{2\sin(x)\cos(x)}{(x+1)^2} - \frac{2\sin^2(x)}{(x+1)^3}$

(D) $\frac{\cos^2(x)}{(x+1)^2}$ (E) $\frac{\cos^2(x)(x+1) - \sin(x)}{(x+1)^4}$

Solution:

(a)(10 pts) Note, by Chain Rule, we have

$$y' = \sec^2(x - \cos(x^2)) \cdot [1 + \sin(x^2) \cdot 2x] = \boxed{\sec^2(x - \cos(x^2))(1 + 2x\sin(x^2))}$$

(b)(10 pts) Taking the derivative of both sides and letting $x = 0$ yields

$$g(x) + xg'(x) + 2g''(x) = 2x \Rightarrow g(0) + 2g''(0) = 0 \Rightarrow g''(0) = -g(0)/2 = -(-10)/2 = \boxed{5}$$

(c)(5 pts) $\boxed{\text{Choice (C).}}$ Note that

$$y' = 2\left[\frac{\sin x}{x+1}\right] \cdot \frac{\cos(x)(x+1) - \sin(x)}{(x+1)^2} = \frac{2\sin(x)\cos(x)(x+1) - 2\sin^2(x)}{(x+1)^3} = \boxed{\frac{2\sin(x)\cos(x)}{(x+1)^2} - \frac{2\sin^2(x)}{(x+1)^3}}$$

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2. (a)(10pts) Suppose that $3 \le f'(x) \le 5$ for all values of x. Show that $18 \le f(8) - f(2) \le 30$. *Hint:* This problem can be done using one of the theorems we studied. Be sure to state the theorem you are using and justify why it applies. Show all work.

(b)(15pts) Ralphie is observing balloon launches with a telescope. Suppose a ballon is rising 400 feet from where Ralphie is standing. If the balloon rises at a rate of 20 feet per minute, how fast is the angle between the ground and the telescope changing when the balloon is 300 feet high? (See diagram below.)

Solution:

(a)(10 pts) Note that since $f'(x)$ exists for all values of x, $f(x)$ is differentiable and therefore continuous for all values of x. In particular $f(x)$ is *continuous* on $[2, 8]$ and *differentiable* on $(2, 8)$ and so by the *Mean Value Theorem* there exists a number $2 < c < 8$ such that

$$f'(c) = \frac{f(8) - f(2)}{8 - 2} = \frac{f(8) - f(2)}{6} \text{ but } 3 \le f'(c) \le 5 \text{ implies } 3 \le \frac{f(8) - f(2)}{6} \le 5 \Rightarrow 18 \le f(8) - f(2) \le 30.$$

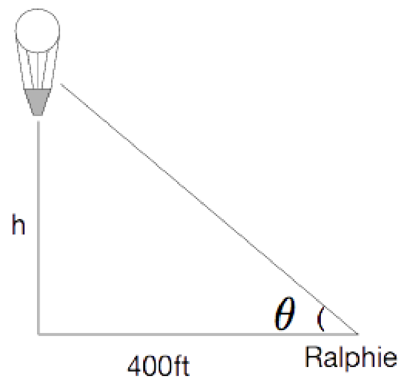
(b)(15 pts) Note that we wish to find $d\theta/dt$ when $h = 300$ ft given that $dh/dt = 20$. From the diagram we see that

$$\tan(\theta) = \frac{h}{400} \Rightarrow \sec^2(\theta)\frac{d\theta}{dt} = \frac{1}{400}\frac{dh}{dt} \Rightarrow \frac{d\theta}{dt} = \frac{\cos^2(\theta)}{400}\frac{dh}{dt}$$

Now note that when $h = 300$, we have that $\cos(\theta) = 400/500 = 4/5$ and using the fact that $dh/dt = 20$, we have

$$\frac{d\theta}{dt} = \frac{\cos^2(\theta)}{400}\frac{dh}{dt} = \frac{20}{400} \cdot \left(\frac{4}{5}\right)^2 = \frac{1}{20} \cdot \left(\frac{4}{5}\right)^2 = \frac{4}{125} \text{ rad/min}$$

so $\boxed{\text{the angle between the ground and the telescope is } \underline{\text{increasing}} \text{ at a rate of } 4/125 \text{ rad/min.}}$



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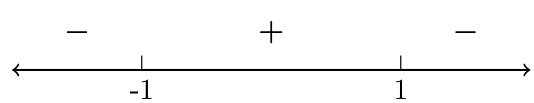
3. (a)(10pts) Find the absolute minimum and maximum values of $f(x) = x\sqrt{4 - x^2}$ for $-1 \leq x \leq 2$. (Be sure to write down the x-coordinate and the y-coordinate of all absolute extrema.)

(b)(10pts) Find the local maximum and minimum values of $g(x) = \dfrac{x}{x^2 + 1}$ if $g'(x) = \dfrac{1 - x^2}{(x^2 + 1)^2}$. Justify your answer with a derivative test. (Be sure to write down the x-coordinate and the y-coordinate of all local extrema.)

(c)(5pts) In your blue book clearly sketch the graph of a function $h(x)$ that satisfies all the following properties (label all extrema, inflection points and asymptotes):

- $h(x)$ is a polynomial and an odd function, $h(0) = 0$ and $h(1) = 2$,
- $\lim_{x \to -0.5^+} h(x) = -1.5$, $\lim_{x \to -\infty} h(x) = +\infty$, and $\lim_{x \to \infty} h(x) = -\infty$,
- The 1st derivative test for $h(x)$ is shown below:

Sign chart for $h'(x)$



- $h''(x) < 0$ on $(-1/2, 0)$ and $(1/2, \infty)$.

Solution:

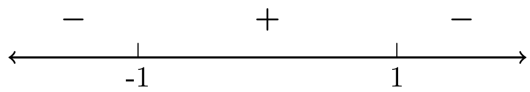
(a)(10 pts) Note that using the product rule, we have

$$f'(x) = (4 - x^2)^{1/2} + x \cdot \frac{1}{2} \cdot (4 - x^2)^{-1/2} \cdot (-2x) = \frac{4 - x^2}{(4 - x^2)^{1/2}} - \frac{x^2}{(4 - x^2)^{1/2}} = \frac{4 - 2x^2}{\sqrt{4 - x^2}}$$

and so $f'(x) = 0$ if $x = \pm\sqrt{2}$ (but $x = -\sqrt{2}$ is not in the given interval) and $f'(x)$ is undefined if $x = \pm 2$ (but $x = -2$ is not in the given interval). Checking the values of $f(x)$ at the endpoints and critical points yields $f(-1) = -\sqrt{3}$, $f(\sqrt{2}) = 2$ and $f(2) = 0$ and so the $\boxed{\text{absolute minimum occurs at } (-1, -\sqrt{3})}$ and the $\boxed{\text{absolute maximum occurs at } (\sqrt{2}, 2).}$

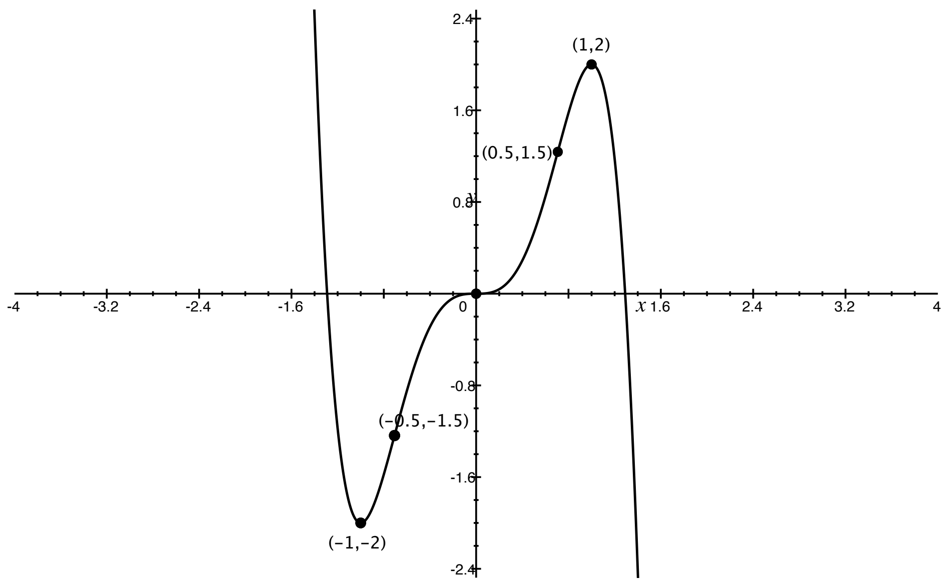
(b)(10 pts) Note that $g'(x) = 0$ if $x = \pm 1$ and $g'(x)$ is defined for all x. Now checking the signs of the derivative about the critical points yields

Sign chart for $g'(x)$



thus we have a $\boxed{\text{local minimum at } (-1, f(-1)) = (-1, -1/2)}$ and a $\boxed{\text{local maximum at } (1, f(1)) = (1, 1/2)}$ by the *1st Derivative Test.*

(c)(5 pts) Note that since $h(x)$ is a polynomial, it is continuous and differentiable for all x. There are no asymptotes, there is a local min at $(-1, -2)$ and a local max at $(1, 2)$ and inflection points at $(-0.5, -1.5)$ and $(0.5, 1.5)$, the graph looks something like:



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4. (a)(15 pts) The position function of a particle is given by $s(t) = t^3 - 3t^2$ where t is in seconds and distance is in feet. (i) When is the particle at rest? (ii) What is the *total distance* traveled by the particle in the first 4 seconds?

(b)(10 pts) Use differentials to estimate the allowable *percentage error* in measuring the radius r of a sphere if the volume of the sphere is to be calculated correctly to within 6% . (Note: The volume is $V = \frac{4}{3}\pi r^3$.)

Solution:

(a)(i)(10 pts) Note that $v(t) = s'(t) = 3t^2 - 6t = 3t(t-2)$ and so $v(t) = 0$ implies that the particle is at rest at $\boxed{t = 0 \text{ seconds and } t = 2 \text{ seconds.}}$

(a)(i)(5 pts) The total distance travelled for $0 \le t \le 4$ is

$$\text{Total Distance} = |s(2) - s(0)| + |s(4) - s(2)| = |-4| + |16 - (-4)| = \boxed{24 \text{ feet}}$$

(b)(10 pts) If the true radius is r and the measure radius is $r + \Delta r$ then we want

$$\frac{V(r+\Delta r) - V(r)}{V(r)} = \frac{\Delta V}{V} \approx \frac{dV}{V} \leq .06 \Rightarrow \frac{4/3 \cdot \pi \cdot 3r^2 dr}{4/3 \cdot \pi r^3} \Rightarrow \frac{3dr}{r} \leq .06 \Rightarrow \frac{dr}{r} \leq .02$$

so the percent error in the measurement of the radius should be at most 2%.