1. (30 pts) Consider the function \( f(x) = \frac{\sqrt{6x + 1} - 5}{x - 4} \).

(a) (10 pts) Give the domain of the function \( y = \frac{1}{f(x)} \) in interval notation.

(b) (10 pts) Find \( \lim_{x \to 0^+} f(x) \). Show all work.

(c) (10 pts) Classify all discontinuities of \( f(x) \). If any of the discontinuities are removable, how would you redefine \( f(x) \) as a piece-wise defined function at the discontinuity to make the function continuous there?

Solution: (a) (10 pts) Note \( y = \frac{x - 4}{\sqrt{6x + 1} - 5} \) and we need \( 6x + 1 \geq 0 \) which implies \( x \geq -1/6 \) and we need \( \sqrt{6x + 1} - 5 \neq 0 \) which implies \( x \neq 4 \), thus the domain is \([-1/6, 4) \cup (4, +\infty)\).

(b) (10 pts) Since \( f(x) \) is continuous at \( x = 0 \) we have \( \lim_{x \to 0^+} f(x) = f(0) = 4/4 = 1 \).

(c) (10 pts) The function \( f(x) \) is undefined for \( x < -1/6 \) and at \( x = 4 \), furthermore at \( x = 4 \) we have \( \lim_{x \to 4^-} \frac{\sqrt{6x + 1} - 5}{x - 4} = \frac{6}{10} = 3/5 \).

thus there is a removable discontinuity at \( x = 4 \) and we can define \( f(x) = \begin{cases} \frac{\sqrt{6x + 1} - 5}{x - 4}, & \text{if } x \neq 4 \\ 3/5, & \text{if } x = 4 \end{cases} \).

2. (24 pts) The following problems are not related. Justify all answers.

(a) (8 pts) Find all vertical asymptotes for the function \( f(x) = \frac{3x^2 - x - 4}{x^2 - 1} \). Justify your answer with limits.

(b) (8 pts) If \( g(x) = \frac{(x + 5)|x + 2|}{(x + 2)} \), find \( \lim_{x \to -2^-} g(x) \) and \( \lim_{x \to -2^+} g(x) \). Does \( \lim_{x \to -2} g(x) \) exist? Why or why not?

(c) (8 pts) Find all horizontal asymptotes of \( h(x) = \begin{cases} x^2/(x^2 + 1), & \text{if } x < 0 \\ x - 2, & \text{if } x \geq 0 \end{cases} \). Justify your answer with limits.

Solution: (a) (8 pts) Note that \( \frac{3x^2 - x - 4}{x^2 - 1} = \frac{(3x - 4)(x + 1)}{(x - 1)(x + 1)} = \frac{3x - 4}{x - 1} \), thus we have a removable discontinuity at \( x = -1 \), not a vertical asymptote and at \( x = 1 \) we have \( \lim_{x \to 1^-} \frac{3x^2 - x - 4}{x^2 - 1} = \lim_{x \to 1^-} \frac{3x - 4}{x - 1} = +\infty \).
and so we have a vertical asymptote at $x = 1$ (note that similarly $\lim_{x \to 1^+} f(x) = -\infty$).

(b)(8 pts) Checking the one-sided limits at $x = -2$ yields

$$\lim_{x \to -2^-} g(x) = \lim_{x \to -2^-} \frac{(x + 5)|x + 2|}{(x + 2)} = \lim_{x \to -2^-} \frac{(x + 5) \cdot [-1]}{(x + 2)} = \lim_{x \to -2^-} -(x + 5) = -3$$

and

$$\lim_{x \to -2^+} g(x) = \lim_{x \to -2^+} \frac{(x + 5)|x + 2|}{(x + 2)} = \lim_{x \to -2^+} \frac{(x + 5) \cdot (x + 2)}{(x + 2)} = \lim_{x \to -2^+} (x + 5) = 3$$

thus $\lim_{x \to -2^-} g(x)$ does not exist since $\lim_{x \to -2^-} g(x) \neq \lim_{x \to -2^+} g(x)$.

(c)(8 pts) For horizontal asymptotes note that

$$\lim_{x \to -\infty} h(x) = \lim_{x \to -\infty} \frac{x^2}{x^2 + 1} = \lim_{x \to -\infty} \frac{x^2}{x^2(1 + 1/x^2)} = 1 \Rightarrow \text{H.A. at } y = 1$$

and

$$\lim_{x \to \infty} h(x) = \lim_{x \to \infty} \frac{x - 2}{x^2 - 4} = \lim_{x \to -\infty} \frac{(x - 2) / (x^2)}{x^2(1 - 4/x^2)} = \lim_{x \to \infty} \frac{(1 - 2/x) / x^2}{(1 - 4/x^2) x^2} = \lim_{x \to \infty} \frac{1 - 2/x}{x(1 - 4/x^2)} = 0 \Rightarrow \text{H.A. at } y = 0$$

so $h(x)$ has horizontal asymptotes at $y = 0$ and $y = 1$.

3. (20 pts) Justify all answers.

(a)(10 pts) Show that the curves $s(x) = \cos(x)$ and $t(x) = x^2 - 2$ intersect at least once in the interval $[0, \pi]$. Note: A graph is not sufficient proof for this problem. (Hint: This can be done using one of the theorems we studied.)

(b)(10 pts) Write the function $f(x) = \frac{x + 4}{|x| + 2}$ as a piece-wise defined function without the absolute value symbol. Is $f(x)$ continuous at $x = 0$? Explain.

**Solution:** (a)(10 pts) Define $f(x) = s(x) - t(x) = \cos(x) - x^2 + 2$ and now we show that $f(x)$ has a root in $[0, \pi]$. Since $f(x)$ is continuous on $[0, \pi]$ and $f(0) = 1 - 0 + 2 = 3 > 0$ and $f(\pi) = -1 - \pi^2 + 2 = 1 - \pi^2 < 0$, i.e. $f(\pi) < 0 < f(0)$, thus, by the Intermediate Value Theorem, there exists at least one number $c$ in $(0, \pi)$ such that $f(c) = 0$. Now if $f(c) = 0$ then $s(c) - t(c) = 0$, i.e. $\cos(c) = c^2 - 2$ for some $c$ in $[0, \pi]$ and so we have shown that $s(x)$ and $t(x)$ intersect at least once in the interval $[0, \pi]$.

(b)(10 pts) Note that $f(x) = \begin{cases} 
\frac{x + 4}{x + 2}, & \text{if } x \geq 0 \\
\frac{x + 4}{2 - x}, & \text{if } x < 0 
\end{cases}$ and note that $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = 4/2 = 2 = f(0)$ and so we see that $f(x)$ is continuous at $x = 0$. 
4. (26 pts) Justify all answers.

(a) (10 pts) A ball thrown vertically upward from the ground at a velocity of 96 ft/sec reaches a height of \( s(t) = -16t^2 + 96t \) feet in \( t \) seconds. Find the instantaneous velocity of the ball at any time \( t \), \( v(t) \), using the limit definition of the derivative.

(b) (8 pts) Find the following limit: \( \lim_{x \to 0} \frac{\tan(x)}{x} \) (You may not use L'Hôpital's Rule.)

(c) (8 pts) Suppose \( f'(x) = \sec(x) \) and \( f(\pi/4) = 1 \), evaluate \( \lim_{x \to \pi/4} \frac{f(x) - 1}{x - \pi/4} \).

**Solution:**

(a) (10 pts) Using the limit definition we have

\[
v(t) = s'(t) = \lim_{h \to 0} \frac{s(t + h) - s(t)}{h} = \lim_{h \to 0} \frac{-16(t + h)^2 + 96(t + h) - (-16t^2 + 96t)}{h}
\]

\[
= \lim_{h \to 0} \frac{-16(t^2 + 2th + h^2) + 96t + 96h - (-16t^2 + 96t)}{h}
\]

\[
= \lim_{h \to 0} \frac{-32th - 16h^2 + 96h}{h} = \lim_{h \to 0} \frac{h(-32t - 16h + 96)}{h} = -32t + 96
\]

so the instantaneous velocity is \( v(t) = -32t + 96 \) ft/sec.

Alternately, one could also calculate

\[
v(t) = s'(t) = \lim_{a \to t} \frac{s(t) - s(a)}{t - a} = \lim_{a \to t} \frac{-16t^2 + 96t - (-16a^2 + 96a)}{t - a}
\]

\[
= \lim_{a \to t} \frac{-16(t^2 - a^2) + 96(t - a)}{t - a}
\]

\[
= \lim_{a \to t} \frac{-16(t-a)(t+a) + 96(t-a)}{t-a} = \lim_{a \to t} -16(t + a) + 96 = -32t + 96.
\]

(b) (8 pts) Note that

\[
\lim_{x \to 0} \frac{\tan(x)}{x} = \lim_{x \to 0} \frac{\sin(x)}{x \cos(x)} = \lim_{x \to 0} \frac{\sin(x)}{x} \cdot \frac{1}{\cos(x)} = 1 \cdot 1 = 1
\]

where in the last equality we used the special limit \( \lim_{x \to 0} \frac{\sin(x)}{x} = 1 \).

(c) (8 pts) By definition we have

\[
\lim_{x \to \pi/4} \frac{f(x) - 1}{x - \pi/4} = \lim_{x \to \pi/4} \frac{f(x) - f(\pi/4)}{x - \pi/4} = f'(\pi/4) = \sec(\pi/4) = \sqrt{2}
\]