1. (30 pts) Consider the function $f(x) = \dfrac{\sqrt{6x+1}-5}{x-4}$.

(a)(10 pts) Give the domain of the function $y = 1/f(x)$ in interval notation.

(b)(10 pts) Find $\lim\limits_{x\to 0^+} f(x)$. Show all work.

(c)(10 pts) Classify all discontinuities of $f(x)$. If any of the discontinuities are removable, how would you redefine $f(x)$ as a *piece-wise* defined function at the discontinuity to make the function continuous there?

Solution: (a)(10 pts) Note $y = \dfrac{x-4}{\sqrt{6x+1}-5}$ and we need $6x+1 \geq 0$ which implies $x \geq -1/6$ and we need $\sqrt{6x+1}-5 \neq 0$ which implies $x \neq 4$, thus the domain is $[-1/6, 4) \cup (4, +\infty)$.

(b)(10 pts) Since $f(x)$ is continuous at $x = 0$ we have $\lim\limits_{x\to 0^+} f(x) = f(0) = 4/4 = 1$.

(c)(10 pts) The function $f(x)$ is undefined for $x < -1/6$ and at $x = 4$, furthermore at $x = 4$ we have

$$\begin{aligned}\lim_{x\to 4} \frac{\sqrt{6x+1}-5}{x-4} &\overset{\text{“0/0”}}{=} \lim_{x\to 4} \frac{\sqrt{6x+1}-5}{x-4} \cdot \frac{\sqrt{6x+1}+5}{\sqrt{6x+1}+5} \\ &= \lim_{x\to 4} \frac{(6x+1)-25}{(x-4)[\sqrt{6x+1}+5]} = \lim_{x\to 4} \frac{6\cancel{(x-4)}}{\cancel{(x-4)}[\sqrt{6x+1}+5]} = 6/10 = 3/5.\end{aligned}$$

thus there is a removable discontinuity at $x = 4$ and we can define $f(x) = \begin{cases} \dfrac{\sqrt{6x+1}-5}{x-4}, & \text{if } x \neq 4 \\ 3/5, & \text{if } x = 4. \end{cases}$

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2. (24 pts) The following problems are not related. Justify all answers.

(a)(8 pts) Find all vertical asymptotes for the function $f(x) = \dfrac{3x^2-x-4}{x^2-1}$. Justify your answer with limits.

(b)(8 pts) If $g(x) = \dfrac{(x+5)|x+2|}{(x+2)}$, find $\lim\limits_{x\to -2^-} g(x)$ and $\lim\limits_{x\to -2^+} g(x)$. Does $\lim\limits_{x\to -2} g(x)$ exist? Why or why not?

(c)(8 pts) Find all horizontal asymptotes of $h(x) = \begin{cases} x^2/(x^2+1), & \text{if } x < 0 \\ \dfrac{x-2}{x^2-4}, & \text{if } x \geq 0 \end{cases}$. Justify your answer with limits.

Solution: (a)(8 pts) Note that $\dfrac{3x^2-x-4}{x^2-1} = \dfrac{(3x-4)\cancel{(x+1)}}{(x-1)\cancel{(x+1)}} = \dfrac{3x-4}{x-1}$, thus we have a removable discontinuity at $x = -1$, not a vertical asymptote and at $x = 1$ we have

$$\lim_{x\to 1^-} \frac{3x^2-x-4}{x^2-1} = \lim_{x\to 1^-} \frac{3x-4}{x-1} = +\infty$$

and so we have a vertical asymptote at $x = 1$ (note that similarly $\lim_{x\to 1^+} f(x) = -\infty$).

(b)(8 pts) Checking the one-sided limits at $x = -2$ yields

$$\lim_{x\to -2^-} g(x) = \lim_{x\to -2^-} \frac{(x+5)|x+2|}{(x+2)} = \lim_{x\to -2^-} \frac{(x+5)\cdot[-(x+2)]}{(x+2)} = \lim_{x\to -2^-} -(x+5) = -3$$

and

$$\lim_{x\to -2^+} g(x) = \lim_{x\to -2^+} \frac{(x+5)|x+2|}{(x+2)} = \lim_{x\to -2^+} \frac{(x+5)\cdot(x+2)}{(x+2)} = \lim_{x\to -2^+} (x+5) = 3$$

thus $\lim_{x\to -2} g(x)$ does not exist since $\lim_{x\to -2^-} g(x) \neq \lim_{x\to -2^+} g(x)$.

(c)(8 pts) For horizontal asymptotes note that

$$\lim_{x\to -\infty} h(x) = \lim_{x\to -\infty} \frac{x^2}{x^2+1} = \lim_{x\to -\infty} \frac{\cancel{x^2}}{\cancel{x^2}(1+1/x^2)} = 1 \Rightarrow \text{ H.A. at } y = 1$$

and

$$\lim_{x\to \infty} h(x) = \lim_{x\to \infty} \frac{x-2}{x^2-4} = \lim_{x\to \infty} \frac{x(1-2/x)}{x^2(1-4/x^2)} = \lim_{x\to \infty} \frac{\cancel{x}(1-2/x)}{x^{\cancel{2}}(1-4/x^2)} = \lim_{x\to \infty} \frac{1-2/x}{x(1-4/x^2)} = 0 \Rightarrow \text{ H.A. at } y = 0$$

so $h(x)$ has horizontal asymptotes at $y = 0$ and $y = 1$.

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3. (20 pts) Justify all answers.

(a)(10 pts) Show that the curves $s(x) = \cos(x)$ and $t(x) = x^2 - 2$ intersect at least once in the interval $[0, \pi]$. Note: A graph is not sufficient proof for this problem. (*Hint:* This can be done using one of the theorems we studied.)

(b)(10 pts) Write the function $f(x) = \dfrac{x+4}{|x|+2}$ as a *piece-wise* defined function without the absolute value symbol. Is $f(x)$ continuous at $x = 0$? Explain.

Solution: (a)(10 pts) Define $f(x) = s(x) - t(x) = \cos(x) - x^2 + 2$ and now we show that $f(x)$ has a root in $[0, \pi]$. Since $f(x)$ is continuous on $[0, \pi]$ and $f(0) = 1 - 0 + 2 = 3 > 0$ and $f(\pi) = -1 - \pi^2 + 2 = 1 - \pi^2 < 0$, *i.e.* $f(\pi) < 0 < f(0)$, thus, by the Intermediate Value Theorem, there exists at least one number c in $(0, \pi)$ such that $f(c) = 0$. Now if $f(c) = 0$ then $s(c) - t(c) = 0$, *i.e.* $\cos(c) = c^2 - 2$ for some c in $[0, \pi]$ and so we have shown that $s(x)$ and $t(x)$ intersect at least once in the interval $[0, \pi]$.

(b)(10 pts) Note that $f(x) = \begin{cases} \dfrac{x+4}{x+2}, & \text{if } x \geq 0 \\ \dfrac{x+4}{2-x}, & \text{if } x < 0 \end{cases}$ and note that $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = 4/2 = 2 = f(0)$ and so we see that $f(x)$ is continuous at $x = 0$.

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4. (26 pts) Justify all answers.

(a)(10 pts) A ball thrown vertically upward from the ground at a velocity of 96 ft/sec reaches a height of $s(t) = -16t^2 + 96t$ feet in t seconds. Find the instantaneous velocity of the ball at any time t, $v(t)$, using the limit definition of the derivative.

(b)(8 pts) Find the following limit: $\lim_{x\to 0} \frac{\tan(x)}{x}$ (You may not use L'Hôpitals Rule.)

(c)(8 pts) Suppose $f'(x) = \sec(x)$ and $f(\pi/4) = 1$, evaluate $\lim_{x\to\pi/4} \frac{f(x)-1}{x-\pi/4}$.

Solution: (a)(10 pts) Using the limit definition we have

$$
\begin{aligned}
v(t) = s'(t) = \lim_{h\to 0} \frac{s(t+h)-s(t)}{h} &= \lim_{h\to 0} \frac{-16(t+h)^2 + 96(t+h) - (-16t^2+96t)}{h} \\
&= \lim_{h\to 0} \frac{-16(t^2+2th+h^2) + 96t + 96h - (-16t^2+96t)}{h} \\
&= \lim_{h\to 0} \frac{-32th - 16h^2 + 96h}{h} = \lim_{h\to 0} \frac{\cancel{h}(-32t-16h+96)}{\cancel{h}} = -32t+96
\end{aligned}
$$

so the instantaneous velocity is $v(t) = -32t + 96$ ft/sec.

Alternately, one could also calculate

$$
\begin{aligned}
v(t) = s'(t) = \lim_{a\to t} \frac{s(t)-s(a)}{t-a} &= \lim_{a\to t} \frac{-16t^2+96t-(-16a^2+96a)}{t-a} \\
&= \lim_{a\to t} \frac{-16(t^2-a^2)+96(t-a)}{t-a} \\
&= \lim_{a\to t} \frac{-16\cancel{(t-a)}(t+a)+96\cancel{(t-a)}}{\cancel{t-a}} = \lim_{a\to t} -16(t+a)+96 = -32t+96.
\end{aligned}
$$

(b)(8 pts) Note that

$$
\lim_{x\to 0} \frac{\tan(x)}{x} = \lim_{x\to 0} \frac{\sin(x)/\cos(x)}{x} = \lim_{x\to 0} \frac{\sin(x)}{x} \cdot \frac{1}{\cos(x)} = 1 \cdot 1 = 1
$$

where in the last equality we used the special limit $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$.

(c)(8 pts) By definition we have

$$
\lim_{x\to\pi/4} \frac{f(x)-1}{x-\pi/4} = \lim_{x\to\pi/4} \frac{f(x)-f(\pi/4)}{x-\pi/4} = f'(\pi/4) = \sec(\pi/4) = \sqrt{2}
$$