1. (26 pts) Evaluate the following.

   (a) Let $g(x) = \cos^3(\pi - 2x)$. Find $g'(\pi/3)$.

   (b) Let $y = (\sqrt{x})^x$. Find $y'(4)$.

   (c) $\lim_{r\to 0^+} e^{-1/r}\ln(r)$

Solution:

(a) (9 pt) $g'(x) = 3\cos^2(\pi-2x)(-\sin(\pi-2x))(-2) = 6\cos^2(\pi-2x)\sin(\pi-2x)$

$g'(\pi/3) = 6\cos^2(\pi/3)\sin(\pi/3) = 6\left(\frac{1}{2}\right)^2\left(\frac{\sqrt{3}}{2}\right) = \boxed{\frac{3\sqrt{3}}{4}}$

(b) (9 pt) Use logarithmic differentiation.

$$y = \left(\sqrt{x}\right)^x$$
$$\ln y = \ln\left(\sqrt{x}\right)^x = x\ln(\sqrt{x}) = \frac{1}{2}x\ln x$$
$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{x}{2}\cdot\frac{1}{x} + \frac{1}{2}\ln x = \frac{1}{2} + \frac{1}{2}\ln x$$
$$\frac{dy}{dx} = \left(\sqrt{x}\right)^x\left(\frac{1}{2} + \frac{1}{2}\ln x\right)$$
$$y'(4) = \left(2^4\right)\left(\frac{1}{2} + \frac{1}{2}\ln 4\right) = \boxed{8(1+\ln 4)} = 8(1 + 2\ln 2)$$

(c) (8 pt) $\lim_{r\to 0^+} e^{-1/r}\ln(r) = \lim_{r\to 0^+}\frac{\ln r}{e^{1/r}} \stackrel{LH}{=} \lim_{r\to 0^+}\frac{1/r}{e^{1/r}(-1/r^2)} = \lim_{r\to 0^+}\frac{-r}{e^{1/r}} = \boxed{0}$
because the numerator approaches 0 and the denominator approaches ∞ as $r \to 0^+$.

2. (24 pts) Evaluate the following integrals.

   (a) $\int \frac{3x^3+5x+7}{x^2}\,dx$ (b) $\int_{\pi^2/16}^{\pi^2/4}\frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$ (c) $\int(\tanh x)\ln(\cosh x)\,dx$

Solution:

(a) (8 pt) $\int \frac{3x^3+5x+7}{x^2}\,dx = \int\left(3x + \frac{5}{x} + \frac{7}{x^2}\right)dx = \boxed{\frac{3}{2}x^2 + 5\ln|x| - \frac{7}{x} + C}$

(b) (8 pt) Let $u = \sqrt{x},\ \ du = dx/(2\sqrt{x})$. Then $x = \pi^2/16 \Rightarrow u = \pi/4$, and $x = \pi^2/4 \Rightarrow u = \pi/2$.

$$\int_{\pi^2/16}^{\pi^2/4}\frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \int_{\pi/4}^{\pi/2} 2\sin u\,du = -2\cos u\Big]_{\pi/4}^{\pi/2}$$
$$= -2(\cos(\pi/2) - \cos(\pi/4)) = -2\left(0 - \sqrt{2}/2\right) = \boxed{\sqrt{2}}$$

(c) (8 pt) Let $u = \ln(\cosh x), du = (\sinh x/\cosh x)\,dx = \tanh x\,dx$.

$$\int(\tanh x)\ln(\cosh x)\,dx = \int u\,du = \frac{1}{2}u^2 + C = \boxed{\frac{1}{2}(\ln(\cosh x))^2 + C}$$

3. (18 pts) For each of the following unrelated problems, identify the relevant theorem, check that its hypotheses are met, and use the theorem to solve the problem.

   (a) Let $f(x) = (x-1)^2(x+2)(x+3)$. Is there a point $x = a$ such that $f(a) = 1$? Justify your answer.

   (b) If $f(1) = 12$, f' is continuous, and $\int_1^6 f'(x)\,dx = 18$, what is the value of $f(6)$?

   (c) Consider the curve $y = \sqrt{x} + \arccos x$ on $[0, 1/2]$. Suppose m equals the slope of the line connecting the endpoints of the curve on the given interval. Must there exist a value c in $(0, 1/2)$ such that $m = y'(c)$? Justify your answer. (*Note:* It is not necessary to calculate m.)

Solution:

   (a) (6 pt) Because f is continuous, $f(0) = 6 > 1$, and $f(1) = 0 < 1$, by the Intermediate Value Theorem, there is a value a such that $f(a) = 1$.

   (b) (6 pt) By the Net Change Theorem (or Fundamental Theorem of Calculus), since f' is continuous, $\int_1^6 f'(x)\,dx = f(6) - f(1) = 18 \Rightarrow f(6) = 12 + 18 = \boxed{30}$.

   (c) (6 pt) The function y is continuous on $[0, 1/2]$ and differentiable on $(0, 1/2)$. By the Mean Value Theorem there is a number c in $(0, 1/2)$ such that $y'(c) = m$.

4. (12 pts) Let $g(x) = \sqrt{\dfrac{3e^{4x}}{e^x - 1}}$.

   (a) Does g have any vertical asymptotes? Justify your answer using appropriate limit(s).

   (b) Find $g(\ln 2)$ and simplify your answer.

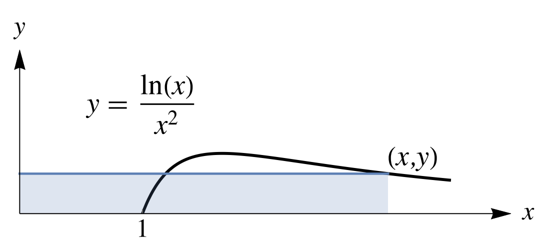
Solution:

   (a) (8 pt) There is a possible V.A. where the denominator equals zero at $x = 0$.

   $\lim_{x\to 0^+} \sqrt{\dfrac{3e^{4x}}{e^x - 1}} = \infty$ because the numerator approaches $\sqrt{3}$ and the denominator approaches 0 with positive values. There is a V.A. at $\boxed{x = 0}$.

   (b) (4 pt) $g(\ln 2) = \sqrt{\dfrac{3e^{4\ln 2}}{e^{\ln 2} - 1}} = \sqrt{\dfrac{3 \cdot 2^4}{2 - 1}} = \boxed{\sqrt{48}} = 4\sqrt{3}$.

5. (12 pts) The rectangle shown has one side on the positive y-axis, one side on the positive x-axis, and its upper right corner on the curve $y = (\ln x)/x^2$, $x \geq 1$. Find the maximum area of the rectangle.



Solution: We wish to maximize the area A of the rectangle.

$$A = xy = x\left(\frac{\ln x}{x^2}\right) = \frac{\ln x}{x}$$

$$A' = \frac{x(1/x) - \ln x}{x^2} = \frac{1-\ln x}{x^2}$$

$$A' = 0 \Rightarrow \frac{1-\ln x}{x^2} = 0 \Rightarrow 1 - \ln x = 0 \Rightarrow x = e.$$

$$A'' = \frac{(x^2)(-1/x) - (1-\ln x)(2x)}{x^4} = \frac{-3+2\ln x}{x^3}$$

Since $A''(e) = -1/e^3$, the curve has a maximum value at the critical number $x = e$. The maximum area of the rectangle is $A(e) = \boxed{1/e}$.

6. (24 pts) The following problems are not related.

   (a) Evaluate $\displaystyle\sum_{k=21}^{100} 2k$.

   (b) Consider the function $g(x) = k \arctan x + b$ where k and b are constants.

      i. Find $g'(1)$.

      ii. If the curve $y = g(x)$ is tangent to the line $y = -2x$ at $x = 1$, find the values of k and b.

   (c) Find the inverse of $y = \dfrac{2-\tan x}{3+\tan x}$, $0 < x < \pi/2$. (You may assume that the function is one-to-one.)

Solution:

(a) (8 pt) $\displaystyle\sum_{k=21}^{100} 2k = 2\sum_{k=1}^{100} k - 2\sum_{k=1}^{20} k = 2\cdot\frac{100\cdot 101}{2} - 2\cdot\frac{20\cdot 21}{2} = 10100 - 420 = \boxed{9680}$.

(b) (8 pt)

   i. $g'(x) = k/(1+x^2) \Rightarrow g'(1) = \boxed{k/2}$.

   ii. The slope of the tangent line equals $g'(1) = k/2 = -2$, so $k = \boxed{-4}$.
   The point of tangency is $(1, -2)$ so $g(1) = k\arctan 1 + b = -2 \Rightarrow$
   $-4(\pi/4) + b = -2 \Rightarrow b = \boxed{\pi - 2}$.

(c) (8 pt)

$$y = \frac{2-\tan x}{3+\tan x}$$

$$3y + y\tan x = 2 - \tan x$$

$$\tan x + y\tan x = 2 - 3y$$

$$\tan x = \frac{2-3y}{1+y}$$

$$x = \arctan\left(\frac{2-3y}{1+y}\right)$$

$$y^{-1} = \boxed{\arctan\left(\frac{2-3x}{1+x}\right)}$$

7. (12 pts) A dose of a painkiller is administered to a patient. Suppose the rate at which the drug is eliminated from the body is proportional to the amount present. If the painkiller has a half-life of seven hours, how long will it take for 90% of the drug to be eliminated?

Solution:

Let $m(t) = m_0e^{kt}$ represent the amount of painkiller in the body t hours after administration. Use the half-life to find k.

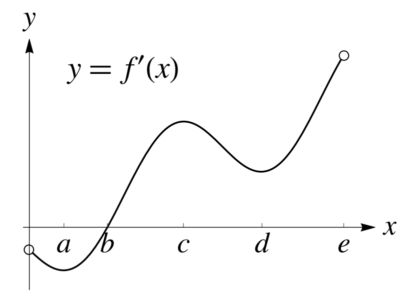
$m(t) = m_0e^{kt} \Rightarrow m(7) = m_0e^{7k} = \frac{1}{2}m_0 \Rightarrow e^{7k} = \frac{1}{2} \Rightarrow 7k = \ln(1/2) \Rightarrow k = -(\ln 2)/7.$

Now solve for t when 10% of the drug remains.

$m(t) = m_0e^{kt} = \frac{1}{10}m_0 \Rightarrow e^{kt} = \dfrac{1}{10} \Rightarrow kt = \ln(1/10) \Rightarrow t = -\dfrac{\ln(10)}{k} = \boxed{\dfrac{7\ln 10}{\ln 2} \text{ hrs}}$

≈ 23.3 hrs.

8. (a) (10 pts) The graph of the first derivative f' of a function f is shown at right. Answer the following questions about the function f which is defined on $[0, e]$. (List all answers that apply. Use interval notation where appropriate. No explanation is necessary.)



i. On what intervals is f increasing?

ii. At what values of x does f have a local minimum value?

iii. On what intervals is f concave up?

iv. At what values of x does f have an inflection point?

Solution:

i. (2 pt) $\boxed{(b, e)}$

ii. (2 pt) $x = \boxed{b}$

iii. (3 pt) $\boxed{(a, c), (d, e)}$

iv. (3 pt) $x = \boxed{a, c, d}$

(b) (12 pts) Sketch a graph of a single function $y = g(x)$ with all of the following properties:

- $\lim_{x \to a} g(x) = g(a)$ for all a except $a = 4$
- $\lim_{x \to 4} g(x) = -\infty$
- $g(2) = 4$
- $g(5) = 2$
- $\lim_{h \to 0} \dfrac{g(2 + h) - g(2)}{h} = 5$
- $\int_5^7 g(x)\, dx = 0$

Solution:

Here is one possible solution. The function is continuous for all $x \neq 4$, has a slope of 5 at $x = 2$, approaches $-\infty$ at $x = 4$, and has a net area of 0 on $[5, 7]$.