1. (30 pts) Evaluate and fully simplify all answers.

   (a) $\displaystyle\int \frac{\sqrt[3]{1+\sqrt{t}}}{\sqrt{t}}\,dt$

   (b) $\displaystyle\int_e^{e^2} \frac{2}{x\ln(x^2)}\,dx$

   (c) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left|\sin^3(\theta)+\cos^2(\theta)\sin(\theta)\right|\,d\theta$

   (d) Consider the function $f(x)=\displaystyle\int_1^{x^2}\sqrt{1+t^3}\,dt$. Find $f'(2)$.

**Solution:**

(a) (7 pts) Let $u = 1+\sqrt{t},\;\; du = dt/(2\sqrt{t})$.

$$\int \frac{\sqrt[3]{1+\sqrt{t}}}{\sqrt{t}}\,dt = \int 2u^{1/3}\,du = 2\cdot\frac{3}{4}u^{4/3}+C = \boxed{\frac{3}{2}\left(1+\sqrt{t}\right)^{4/3}+C}.$$

(b) (7 pts) Let $u=\ln x,\;\; du = dx/x$. When $x=e, u=1$ and when $x=e^2, u=2$.

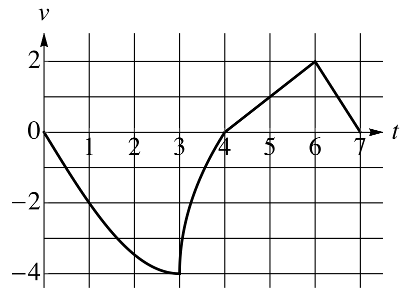
$$\int_e^{e^2}\frac{2}{x\ln(x^2)}\,dx = \int_e^{e^2}\frac{\cancel{2}}{\cancel{2}x\ln(x)}\,dx = \int_1^2\frac{du}{u} = \ln|u|\big]_1^2 = \ln 2-\ln 1 = \boxed{\ln 2}.$$

(c) (8 pts)

$$\begin{aligned}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left|\sin^3(\theta)+\cos^2(\theta)\sin(\theta)\right|\,d\theta &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left|\sin(\theta)\left(\sin^2(\theta)+\cos^2(\theta)\right)\right|\,d\theta\\ &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}|\sin(\theta)|\,d\theta = 2\int_0^{\frac{\pi}{4}}\sin(\theta)\,d\theta\\ &= \left[-2\cos(\theta)\right]_0^{\pi/4} = -2\left(\frac{1}{\sqrt{2}}-1\right) = \boxed{-\sqrt{2}+2}.\end{aligned}$$

(d) (8 pts) By the FTC, $f'(x)=2x\sqrt{1+x^6}$ so $f'(2)=\boxed{4\sqrt{65}}$.

2. (15 pts) The graph below shows the velocity $v$ (in cm/sec) of an object moving up and down for $t = 0$ to 7 sec. (For parts (a) to (c) no explanation is necessary.)




(a) On which interval(s) was the acceleration negative? Express your answer in interval notation.

(b) At what time was the object farthest away from the starting point?

(c) Did the object end up above, below, or at the starting position?

(d) What was the average velocity of the object on $[5, 7]$? Justify your answer.

**Solution:**

(a) (3 pts) $\boxed{(0,3),(6,7)}$

(b) (3 pts) $t = \boxed{4 \text{ sec}}$

(c) (3 pts) $\boxed{\text{below}}$

(d) (6 pts) $v_{ave} = \dfrac{1}{7-5}\displaystyle\int_5^7 v(t)\,dt = \dfrac{1}{2}\left(\dfrac{3}{2}+1\right) = \boxed{\dfrac{5}{4}\text{ cm/sec}}$ using geometry.

3. (16 pts) The following problems are not related.

(a) Express $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^4}{n^5}$ as a definite integral and use it to find the value of the limit.

(b) Evaluate the upper sum for $f(x) = (x-1)^2, -3 \le x \le 3$, with $n = 3$ equal subintervals.

**Solution:**

(a) (8 pts) $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x.$

Let $\Delta x = (b-a)/n = 1/n$ and $x_i = a + i\Delta x = i/n$. Then $a = 0, b = 1, f(x) = x^4$, and

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^4}{n^5} = \int_0^1 x^4\,dx = \frac{x^5}{5}\bigg]_0^1 = \boxed{\frac{1}{5}}.$$

(b) (8 pts) The upper sum is $2(f(-3)+f(-1)+f(3)) = 2(16+4+4) = \boxed{48}$.

4. (24 pts) The following problems are not related.

(a) Write the following expression in sigma notation and find the sum.

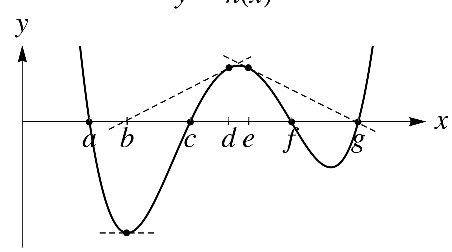
$$(1^3+10)+(2^3+20)+(3^3+30)+\cdots+(20^3+200)$$

(b) Consider the function

$$f(x)=\frac{(1-x)(2+x)^2(3-x/2)}{(4+x)^2}.$$

Use logarithmic differentiation to find $f'(0)$.

(c) Suppose Newton's Method is used to find a root of the function $h(x)$. The tangent lines at $x=b,d,$ and $e$ are shown. (No explanation is necessary for the following questions.)




i. For which of the initial approximations $x_1=a,b,c,d,e,f,$ and $g$ will Newton's Method fail to find a root? List all that apply.

ii. Which of the initial approximations $x_1=b,d,e,$ and $g$ will lead to a second approximation $x_2$ that equals one of the roots of $h$? List all that apply.

**Solution:**

(a) (8 pts)

$$\sum_{k=1}^{20}(k^3+10k)=\sum_{k=1}^{20}k^3+10\sum_{k=1}^{20}k=\left(\frac{20\cdot 21}{2}\right)^2+10\cdot\frac{20\cdot 21}{2}=210^2+10(210)=\boxed{46200}.$$

(b) (8 pts)

$$y=\frac{(1-x)(2+x)^2(3-x/2)}{(4+x)^2}$$

$$\ln y=\ln(1-x)+2\ln(2+x)+\ln(3-x/2)-2\ln(4+x)$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{-1}{1-x}+\frac{2}{2+x}-\frac{1}{2(3-x/2)}-\frac{2}{4+x}$$

$$\frac{dy}{dx}=y\left(\frac{-1}{1-x}+\frac{2}{2+x}-\frac{1}{2(3-x/2)}-\frac{2}{4+x}\right)$$

$$f'(0)=f(0)\left(-1+1-\frac{1}{6}-\frac{1}{2}\right)=\frac{3}{4}\left(-\frac{2}{3}\right)=\boxed{-\frac{1}{2}}$$

(c) (8 pts) i. $\boxed{b,d}$ ii. $\boxed{e,g}$

5. (15 pts) Let $f(x) = \dfrac{5e^x}{4 - e^x}$.

(a) Find the domain of $f$ in interval notation.

(b) Find the inverse function $f^{-1}$. (You may assume that $f$ is one-to-one.)

(c) If $\left(f^{-1}\right)'(5/3) = 1/f'(a)$, what is the value of $a$? (*Hint:* It is not necessary to differentiate the functions to find the answer.)

**Solution:**

(a) (4 pts) $\boxed{(-\infty, \ln 4) \cup (\ln 4, \infty)}$

(b) (7 pts) $y = \dfrac{5e^x}{4 - e^x} \Rightarrow 5e^x = 4y - ye^x \Rightarrow e^x(5 + y) = 4y \Rightarrow e^x = \dfrac{4y}{5 + y} \Rightarrow$

$x = \ln\left(\dfrac{4y}{5 + y}\right)$. The inverse function is $f^{-1}(x) = \boxed{\ln\left(\dfrac{4x}{5 + x}\right)}$.

(c) (4 pts) Since $\left(f^{-1}\right)'(f(a)) = 1/f'(a)$, solve $f(a) = 5/3$.

$f(a) = \dfrac{\cancel{5}e^a}{4 - e^a} = \dfrac{\cancel{5}}{3} \Rightarrow 3e^a = 4 - e^a \Rightarrow e^a = 1 \Rightarrow a = \boxed{0}$.