1. (28 pts) The following problems are not related.

   (a) Let $y = \cos^4\left(3u^2\right)$. Find dy/du.

   (b) Let $x\sqrt{y+4} = y^2 - 4^2$. Find dy/dx at the point $(3, 5)$.

   (c) Suppose that $-2 \le f'(x) \le 10$ for all values of x. Find the smallest and largest possible values of $f(15) - f(6)$.

   (d) Let $y = \dfrac{10x^3 - 6x^2 + 15x}{2x^2 + 3}$. Find an equation for the slant asymptote of y.

   (It is not necessary to evaluate any limits when justifying your answer.)

Solution:

(a) (6 pts)

$$\begin{aligned} y &= \cos^4\left(3u^2\right) \\ \frac{dy}{du} &= 4\cos^3\left(3u^2\right)\left(-\sin(3u^2)\right)(6u) \\ &= \boxed{-24u\cos^3\left(3u^2\right)\sin(3u^2)} \end{aligned}$$

(b) (10 pts) Use implicit differentiation.

$$\begin{aligned} x\sqrt{y+4} &= y^2 - 4^2 \\ x \cdot \frac{1}{2\sqrt{y+4}} \cdot \frac{dy}{dx} + \sqrt{y+4} &= 2y\frac{dy}{dx} \\ \sqrt{y+4} &= \frac{dy}{dx}\left(2y - \frac{x}{2\sqrt{y+4}}\right) \\ \frac{dy}{dx} &= \frac{\sqrt{y+4}}{2y - \frac{x}{2\sqrt{y+4}}} = \frac{2(y+4)}{4y\sqrt{y+4} - x} \\ \left.\frac{dy}{dx}\right|_{(3,5)} &= \frac{\sqrt{9}}{2(5) - \frac{3}{2\sqrt{9}}} = \boxed{\frac{6}{19}}. \end{aligned}$$

(c) (6 pts) By the Mean Value Theorem, $f(15) - f(6) = f'(c)(15-6)$ for some c in $(6, 15)$. Since
$-2 \le f'(c) \le 10$, it follows that $-18 \le 9f'(c) \le 90$ and $\boxed{-18} \le f(15) - f(6) \le \boxed{90}$.
(f is differentiable for all x and therefore continuous for all x.)

(d) (6 pts)

$$
\begin{array}{r|l}
 & \;5x - 3 \\
2x^2 + 3 & 10x^3 - 6x^2 + 15x \\
 & -10x^3 - 15x \\
\hline
 & -6x^2 \\
 & 6x^2 + 9 \\
\hline
 & 9
\end{array}
$$

There is a slant asymptote at $\boxed{y = 5x - 3}$.

2. (15 pts) Beatrice takes a 36π cm^3 lump of clay and shapes it into a cylinder of length L and radius r. As she rolls the clay the cylindrical shape is maintained. How fast is the radius r changing when $L = 9$ cm and L is increasing at a rate of $\frac{1}{2}$ cm/sec?

Solution:

NOTE: This problem is very similar to HW 6 #6.

The clay has a fixed volume of $V = \pi r^2 L = 36\pi$ cm^3. We wish to find dr/dt when $L = 9$ and $dL/dt = 1/2$.

At this moment $r = 2$ cm.

$$
\begin{aligned}
V &= \pi r^2 L \\
\frac{dV}{dt} &= \pi r^2 \frac{dL}{dt} + 2\pi r L \frac{dr}{dt} \\
0 &= \pi(4)\left(\frac{1}{2}\right) + 2\pi(2)(9)\frac{dr}{dt} \\
\frac{dr}{dt} &= \boxed{-\frac{1}{18} \text{ cm/sec}}.
\end{aligned}
$$

3. (27 pts) Consider the function $y = \dfrac{1}{x(x-3)^2}$ with $y' = \dfrac{3(1-x)}{x^2(x-3)^3}$ and $y'' = \dfrac{6\left(2x^2 - 4x + 3\right)}{x^3(x-3)^4}$.

(a) Find the domain of the function in interval notation.

(b) Does the function have vertical asymptotes? Justify your answer using appropriate limits.

(c) Does the function have horizontal asymptotes? Justify your answer using appropriate limits.

(d) On what intervals is y increasing? decreasing?

(e) Find the x and y coordinates of the local maximum and minimum extrema, if any.

(f) On what intervals is y concave up? concave down?

(g) Sketch a graph of y. Clearly label any asymptotes and local extrema.

Solution:

(a) (3 pts) $\boxed{(-\infty, 0) \cup (0, 3) \cup (3, \infty)}$

(b) (4 pts) $\displaystyle\lim_{x\to 0^+} \frac{1}{x(x-3)^2} = \infty$ and $\displaystyle\lim_{x\to 3^+} \frac{1}{x(x-3)^2} = \infty$ so there are vertical asymptotes at $\boxed{x = 0}$ and $\boxed{x = 3}$.

(c) (2 pts) $\displaystyle\lim_{x\to \infty} \frac{1}{x(x-3)^2} = 0$ so there is a horizontal asymptote at $\boxed{y = 0}$.

(d) (6 pts) The first derivative y' equals 0 at $x = 1$ and is undefined at $x = 0$ and 3.

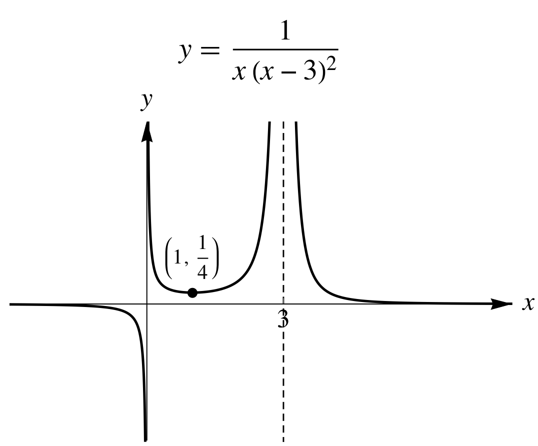
Intervals	y'	y
$x < 0$	–	decreasing on $(-\infty, 0)$
$0 < x < 1$	–	decreasing on $(0, 1)$
$1 < x < 3$	+	increasing on $(1, 3)$
$x > 3$	–	decreasing on $(3, \infty)$

(e) (2 pts) There is a local minimum value at $\boxed{(1, 1/4)}$.

(f) (6 pts) The second derivative y'' does not equal 0 for any x. It is undefined at $x = 0$ and 3.

Intervals	y''	y
$x < 0$	–	concave down on $(-\infty, 0)$
$0 < x < 3$	+	concave up on $(0, 3)$
$x > 3$	+	concave up on $(3, \infty)$

(g) (4 pts)



4. (14 pts) Find the x and y coordinates of the absolute maximum and minimum values (if any) of $f(x) = \dfrac{3\sin(x)}{5+2\sin(x)}$ on $[-\pi, \pi]$.

Solution:

$$f'(x) = \frac{(5+2\sin x)(3\cos x) - 3\sin x(2\cos x)}{(5+2\sin x)^2} = \frac{15\cos x}{(5+2\sin x)^2}.$$

$f'(x) = 0 \Rightarrow 15\cos x = 0 \Rightarrow x = \pm\pi/2.$

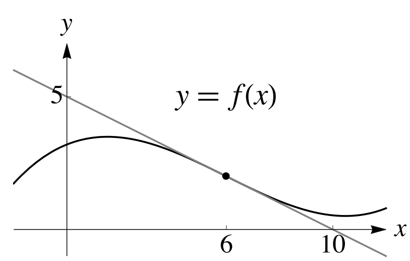
At the critical numbers $f(-\pi/2) = -1$ and $f(\pi/2) = 3/7$.

At the endpoints $f(-\pi) = 0$ and $f(\pi) = 0$.

There is an absolute minimum at $\boxed{(-\pi/2, -1)}$ and an absolute maximum at $\boxed{(\pi/2, 3/7)}$.

5. (16 pts) Shown below is the tangent line to a function $y = f(x)$ at $x = 6$.

   (a) Use the graph to find an equation for the tangent line.
   (b) Find the values of $f(6)$ and $f'(6)$.
   (c) Use a linear approximation to estimate the value of $f(6.02)$.
   (d) Now suppose $f(x)$ is a cubic function $ax^3 + bx^2 + cx + d$ where a, b, c, and d are constants. If the value of $f''(6)$ is 0, find the ratio a/b.



Solution:

(a) (4 pts) The line has a slope of $-1/2$ and a y-intercept of 5. The equation is $\boxed{y = -x/2 + 5}$ or $\boxed{y = 2 - \frac{1}{2}(x-6)}$.

(b) (4 pts) $f(6) = \boxed{2}$ and $f'(6) = \boxed{-1/2}$.

(c) (4 pts) $L(x) = -x/2 + 5 \Rightarrow L(6.02) = -3.01 + 5 = \boxed{1.99}$.

(d) (4 pts) $f'(x) = 3ax^2 + 2bx + c$ and $f''(x) = 6ax + 2b$ so $f''(6) = 36a + 2b = 0 \Rightarrow a/b = \boxed{-1/18}$.