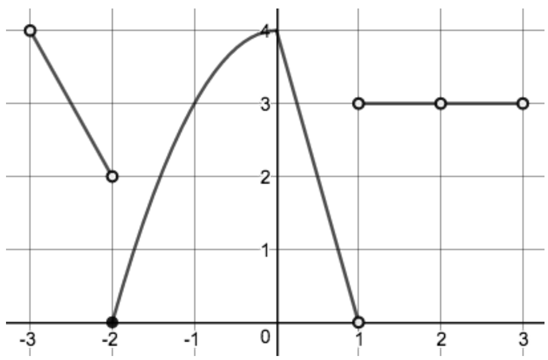
1. (19 pts) Using the function $f(x)$ plotted in the figure, find the following values, or write DNE if they do not exist. No justification is necessary for this problem.

Solution:



(a) $\lim_{x \to -2} f(x) = \boxed{\text{DNE}}$

(b) $\lim_{x \to -2^-} f(x) = \boxed{2}$

(c) $\lim_{x \to -2^+} f(x) = \boxed{0}$

(d) $f(1) = \boxed{\text{DNE}}$

(e) $\lim_{x \to 2} f(x) = \boxed{3}$

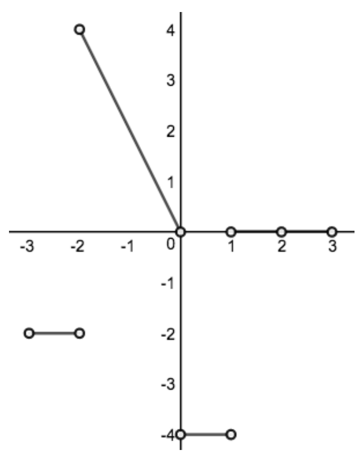
(f) $f'(\frac{3}{5}) = \boxed{-4}$

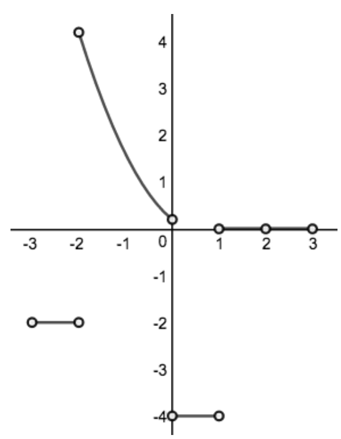
(g) the average rate of change of f on $\left[-2, \frac{1}{2}\right]$ is $\frac{2}{5/2} = \boxed{4/5}$.

(h) the values of x in $(-3, 3)$ where f is not differentiable are $\boxed{-2, 0, 1, 2}$.

2. (10 pts) Use the graph of $f(x)$ shown above to sketch a graph of the derivative $f'(x)$. Label tick marks clearly.

Solution: Here are two possible solutions.





3. (17 pts) The following problems are not related.

(a) If $\tan\theta = 1/x$, $\pi < \theta < \frac{3\pi}{2}$, what is the value of $\csc\theta$ in terms of x?

(b) Find the domain of the function $y = \dfrac{x}{\sqrt{x+2} - 3}$. Express your answer in interval notation.

(c) Let $f(x) = 7 - 2x$. Suppose we use the precise definition of a limit to verify the value of $\lim_{x \to 1/2} (7 - 2x)$ and we find that if $0.4 < x < 0.6$ then $5.8 < f(x) < 6.2$. What are the corresponding values of ϵ and δ?

(d) Let $h(x) = \left|x^3 + 1\right|$.

i. Sketch a graph of $y = h(x)$. Label all intercepts.

ii. Express h as a piecewise-defined function without using absolute value.

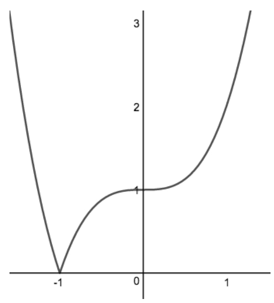
Solution:

(a) (4 pts) The hypotenuse of a right triangle with legs of length 1 and x has length $\sqrt{1+x^2}$. It follows that in Q3, $\csc\theta = \boxed{-\sqrt{1+x^2}}$.

(b) (4 pts) The square root expression is defined for $x \geq -2$. The function is undefined when the denominator equals 0 at $x = 7$. The domain therefore is $\boxed{[-2,7) \cup (7,\infty)}$.

(c) (4 pts) The intervals correspond to $|x - 0.5| < 0.1$ and $|f(x) - 6| < 0.2$, so the values are $\boxed{\epsilon = 0.2 \text{ and } \delta = 0.1}$.

(d) i. (2 pts)



ii. (3 pts) $\boxed{h(x) = \begin{cases} x^3 + 1, & x \geq -1 \\ -x^3 - 1, & x < -1. \end{cases}}$

4. (26 pts)

(a) Does the function $y = \dfrac{\cos^2(x) - \sin^2(x)}{\cos(x) - \sin(x)}$ have a vertical asymptote at $x = \pi/4$? Justify your answer by evaluating appropriate limits.

(b) Does the function $y = \dfrac{6x+2}{\sqrt{4x^2 - 2}}$ have horizontal asymptotes? Justify your answer by evaluating appropriate limits.

(c) Evaluate $\displaystyle\lim_{x \to 0} \frac{x\cot(2x)}{5}$.

Solution:

(a) (8 pts) $\displaystyle\lim_{x \to \pi/4} \frac{\cos^2(x) - \sin^2(x)}{\cos(x) - \sin(x)} = \lim_{x \to \pi/4} \frac{(\cos x + \sin x)\cancel{(\cos x - \sin x)}}{\cancel{\cos x - \sin x}} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}.$

There is $\boxed{\text{no vertical asymptote}}$ at $x = \pi/4$.

(b) (7+3 pts) First check x approaching ∞:

$$\lim_{x\to\infty} \frac{6x+2}{\sqrt{4x^2-2}} = \lim_{x\to\infty} \frac{6x+2}{\sqrt{x^2(4-2/x^2)}} = \lim_{x\to\infty} \frac{6x+2}{|x|\sqrt{4-2/x^2}} = \lim_{x\to\infty} \frac{6x+2}{x\sqrt{4-2/x^2}} \cdot \frac{1/x}{1/x}$$

$$= \lim_{x\to\infty} \frac{6+2/x}{\sqrt{4-2/x^2}} = 3.$$

There is a horizontal asymptote at $\boxed{y=3}$. Now check x approaching $-\infty$:

$$\lim_{x\to-\infty} \frac{6x+2}{\sqrt{4x^2-2}} = \lim_{x\to-\infty} \frac{6x+2}{|x|\sqrt{4-2/x^2}} = \lim_{x\to-\infty} \frac{6x+2}{-x\sqrt{4-2/x^2}} = -3.$$

There is a horizontal asymptote at $\boxed{y=-3}$.

(c) (8 pts) $\displaystyle\lim_{x\to 0} \frac{x\cot(2x)}{5} = \lim_{x\to 0} \frac{x}{5} \cdot \frac{\cos(2x)}{\sin(2x)} \cdot \frac{2}{2} = \lim_{x\to 0} \frac{1}{5} \cdot \frac{2x}{\sin(2x)} \cdot \frac{1}{2} \cdot \cos(2x) = \frac{1}{5} \cdot 1 \cdot \frac{1}{2} \cdot 1 = \boxed{\frac{1}{10}}$.

5. (14 pts) Consider the function $f(x) = \begin{cases} \dfrac{x^2+2x-3}{x-1} & x < -1 \\ c & x = -1 \\ b\cos(\pi x) & x > -1. \end{cases}$

(a) Write the definition of continuity of an arbitrary function f at a number a.

(b) Are there constants b and c that make $f(x)$ continuous at $x = -1$? Use the definition of continuity to find b and c, or to explain why they don't exist.

Solution:

(a) (3 pts) $\boxed{f(a) = \lim_{x\to a} f(x)}$ or $f(a) = \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$.

(b)

(5 pts) $\displaystyle\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \frac{x^2+2x-3}{x-1} = \lim_{x\to -1^-} x+3 = 2$

(3 pts) $\displaystyle\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} b\cos(\pi x) = -b$

$f(-1) = c$

(3 pts) In order for $f(x)$ to be continuous, these values must agree. Thus, $2 = -b = c$. Therefore, $\boxed{b=-2}$ and $\boxed{c=2}$.

6. (14 pts) Let $g(x) = \dfrac{1}{7-2x}$.

(a) Write the limit definition of the derivative of an arbitrary function $f(x)$ at a number a.

(b) Use your answer for part (a) to find the value of $g'(0)$.

(c) Find an equation for the line tangent to $y = g(x)$ at $x = 0$.

Solution:

(a) (3 pts) $\boxed{f'(a) = \lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h}}$ or $\lim_{x\to a} \dfrac{f(x)-f(a)}{x-a}$.

(b) (7 pts)

Solution:

$$g'(0) = \lim_{h\to 0} \frac{g(h)-g(0)}{h} = \lim_{h\to 0} \frac{\frac{1}{7-2h} - \frac{1}{7}}{h} = \lim_{h\to 0} \frac{1}{h} \cdot \frac{7-(7-2h)}{7(7-2h)} = \lim_{h\to 0} \frac{1}{\not h} \cdot \frac{2\not h}{7(7-2h)} = \boxed{\frac{2}{49}}.$$

(c) (4 pts) The tangent slope is $2/49$ and the point of tangency is $(0, 1/7)$ so the tangent line is $\boxed{y - \frac{1}{7} = \frac{2}{49}x}$ or $y = \frac{1}{7} + \frac{2}{49}x$.