1. (25 pts) Evaluate the following integrals.

(a) $\displaystyle\int_0^2 \frac{4x}{(x^2+1)^2}\,dx$ (b) $\displaystyle\int \frac{5^t}{5^t-1}\,dt$ (c) $\displaystyle\int_{-1/2}^0 \frac{6}{\sqrt{1-x^2}}\,dx$

Solution:

(a) (9 pts) Let $u = x^2+1,\ \ du = 2x\,dx$.

$$\int_0^2 \frac{4x}{(x^2+1)^2}\,dx = \int_1^5 \frac{2}{u^2}\,du = 2\left[-\frac{1}{u}\right]_1^5 = 2\left(-\frac{1}{5}+1\right) = \boxed{\frac{8}{5}}$$

(b) (8 pts) Let $u = 5^t - 1,\ \ du = 5^t(\ln 5)\,dt$.

$$\int \frac{5^t}{5^t-1}\,dt = \frac{1}{\ln 5}\int \frac{du}{u} = \frac{1}{\ln 5}\ln|u| + C = \boxed{\frac{\ln\left|5^t-1\right|}{\ln 5} + C} \text{ or } \log_5\left|5^t-1\right| + C$$

(c) (8 pts) $\displaystyle\int_{-1/2}^0 \frac{6}{\sqrt{1-x^2}}\,dx = 6\arcsin x\Big|_{-1/2}^0 = 6(0+\pi/6) = \boxed{\pi}$

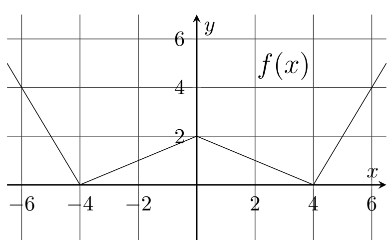
2. (26 pts) The following problems are not related.

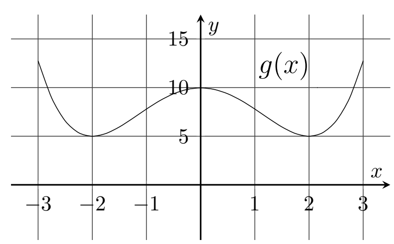
(a) Show that the function $h(x) = \cos x + 2x + 1$ has at least one real root. Indicate an interval where the root can be found.

(b) Find an equation for the line tangent to $y = \sin^3(2x)$ at $x = \pi/6$. Express your answer in the form $y = mx + b$.

(c) Use the graphs of $f(x)$ and $g(x)$, shown below, to find the following values. (No justification is required.)

i. $f'(-3) + f'(1)$ ii. $f'(g(2))$ iii. $(fg)'(2)$





Solution:

(a) (8 pts) By the Intermediate Value Theorem, since

$$h(-\pi/2) = -\pi + 1 < 0, \qquad h(0) = 2 > 0,$$

and h is continuous, there is a root in the interval $\boxed{(-\pi/2, 0)}$.

(b) (9 pts)

$$\begin{aligned} y &= \sin^3(2x) \\ y(\pi/6) &= \sin^3(\pi/3) = \left(\sqrt{3}/2\right)^3 = 3\sqrt{3}/8 \\ y' &= 6\sin^2(2x)\cos(2x) \\ y'(\pi/6) &= 6\sin^2(\pi/3)\cos(2x) = 6 \cdot \frac{3}{4} \cdot \frac{1}{2} = \frac{9}{4} \end{aligned}$$

The tangent line is $y = \dfrac{3\sqrt{3}}{8} + \dfrac{9}{4}(x - \pi/6)$ or $\boxed{y = \dfrac{9}{4}x + \dfrac{3\sqrt{3}}{8} - \dfrac{3\pi}{8}}$.

(c) (9 pts)

i. $f'(-3) + f'(1) = 1/2 - 1/2 = \boxed{0}$

ii. $f'(g(2)) = f'(5) = \boxed{2}$

iii. $(fg)'(2) = f(2)g'(2) + g(2)f'(2) = 1(0) + 5(-1/2) = \boxed{-5/2}$

3. (18 pts)

(a) Find the value of $\lim_{x\to 0} \arctan\left(\frac{1}{x}\right)$.

(b) Use the Squeeze Theorem to find the value of $\lim_{x\to 0} |x| \arctan\left(\frac{1}{x}\right)$.

(c) Does $\lim_{x\to\infty} \dfrac{\arctan(1/x^2)}{\arctan(1/x)}$ exist? Justify your answer.

Solution:

(a) (6 pts) $\lim_{x\to 0} \arctan\left(\frac{1}{x}\right)$ $\boxed{\text{does not exist}}$ because $\lim_{x\to 0^+} \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$ and $\lim_{x\to 0^-} \arctan\left(\frac{1}{x}\right) = -\frac{\pi}{2}$.

(b) (6 pts)

$$\begin{aligned} -\tfrac{\pi}{2} &< \arctan\left(\tfrac{1}{x}\right) < \tfrac{\pi}{2} \\ -\tfrac{\pi}{2}|x| &< |x|\arctan\left(\tfrac{1}{x}\right) < \tfrac{\pi}{2}|x| \end{aligned}$$

Since $\lim_{x\to 0} -\frac{\pi}{2}|x| = \lim_{x\to 0} -\frac{\pi}{2}|x| = 0$, then $\lim_{x\to 0} |x| \arctan\left(\frac{1}{x}\right) = \boxed{0}$.

(c) (6 pts)

$$\lim_{x\to\infty} \frac{\arctan(1/x^2)}{\arctan(1/x)} \overset{LH}{=} \lim_{x\to\infty} \frac{\dfrac{1}{1+\frac{1}{x^4}}\left(\dfrac{-2}{x^3}\right)}{\dfrac{1}{1+\frac{1}{x^2}}\left(\dfrac{-1}{x^2}\right)} = \lim_{x\to\infty} \frac{2x^3 + 2x}{x^4 + 1} = \lim_{x\to\infty} \frac{\dfrac{2}{x} + \dfrac{2}{x^3}}{1 + \dfrac{1}{x^4}} = \boxed{0}$$

4. (16 pts) For this problem let $f(x) = (\ln x)^x$.

   (a) Find the domain of f.

   (b) Find the instantaneous rate of change of f with respect to x.

   (c) Find the value of $\lim_{h\to 0} \dfrac{(\ln(e+h))^{e+h} - 1}{h}$.

Solution:

(a) (4 pts) Domain: $\boxed{x \geq 1}$ or $[1, \infty)$.

(b) (6 pts)

$$\begin{aligned} y &= (\ln x)^x \\ \ln y &= \ln(\ln x)^x \\ \ln y &= x \ln(\ln x) \\ \frac{1}{y}\frac{dy}{dx} &= x \cdot \frac{1}{\ln x} \cdot \frac{1}{x} + \ln(\ln x) \\ \frac{dy}{dx} &= \boxed{(\ln x)^x \left(\frac{1}{\ln x} + \ln(\ln x)\right)} \end{aligned}$$

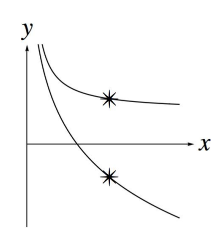
Alternate Solution:

$$y = (\ln x)^x = e^{\ln(\ln x)^x}$$

$$y' = e^{\ln(\ln x)^x} \cdot \frac{d}{dx}\left(\ln(\ln x)^x\right) = (\ln x)^x \left(\frac{1}{\ln x} + \ln(\ln x)\right)$$

(c) (6 pts) By the definition of the derivative, $\lim_{h\to 0} \dfrac{(\ln(e+h))^{e+h} - 1}{h} = f'(e) = 1(1+0) = \boxed{1}$.

5. (20 pts) Bug A is moving along the curve $y = 1 + 1/x$ and Bug B is moving along the curve $y = 1 - 2\ln x$ so that the bugs are always vertically aligned (one directly above the other).



   (a) The distance between the two bugs is minimized at what x-coordinate?

   (b) As Bug A reaches $x = 2$, its y-coordinate is decreasing at a rate of 0.1 unit/sec. How fast is Bug B's y-coordinate changing then?

Solution:

(a) (10 pts) Let d equal the distance between the bugs:

$$d = \left(1 + \frac{1}{x}\right) - (1 - 2\ln x) = \frac{1}{x} + 2\ln x.$$

Then $d' = -\dfrac{1}{x^2} + \dfrac{2}{x} = \dfrac{-1+2x}{x^2}$ and $d' = 0$ at $x = 1/2$.

Since $d'' = \dfrac{2}{x^3} - \dfrac{2}{x^2}$ and $d''(1/2) = 16 - 8 > 0$, the distance is minimized at $x = \boxed{1/2}$.

(b) (10 pts) Let $y_1 = 1 + 1/x$ and $y_2 = 1 - 2\ln x$. We are given that $dy_1/dt = -0.1$ unit/sec when $x = 2$. We wish to find dy_2/dt. First solve for dx/dt.

$$y_1 = 1 + \frac{1}{x} \;\Rightarrow\; \frac{dy_1}{dt} = -\frac{1}{x^2}\frac{dx}{dt} \;\Rightarrow\; -0.1 = -\frac{1}{4}\frac{dx}{dt} \;\Rightarrow\; \frac{dx}{dt} = 0.4 \text{ unit/sec}$$

Bug B's x-coordinate will be changing at the same rate. Solve for dy_2/dt.

$$y_2 = 1 - 2\ln x \;\Rightarrow\; \frac{dy_2}{dt} = -\frac{2}{x}\frac{dx}{dt} \;\Rightarrow\; \frac{dy_2}{dt} = -\frac{2}{2}(0.4) = \boxed{-0.4 \text{ unit/sec}}$$

6. (25 pts) The following problems are not related.

(a) Find all asymptotes (if any) of the function $h(x) = \dfrac{e^x}{2-e^x}$. Justify your answer using limits.

(b) Let $y = x^2\sqrt{5-x}$. Find (i) the domain of the function, and (ii) the intervals of increase and decrease. Express your answers in interval notation.

Solution:

(a) (12 pts) Horizontal asymptotes:

$$\lim_{x\to\infty} \frac{e^x}{2-e^x} \overset{LH}{=} \lim_{x\to\infty} \frac{e^x}{-e^x} = -1 \quad \text{and} \quad \lim_{x\to-\infty} \frac{e^x}{2-e^x} = \frac{0}{2-0} = 0.$$

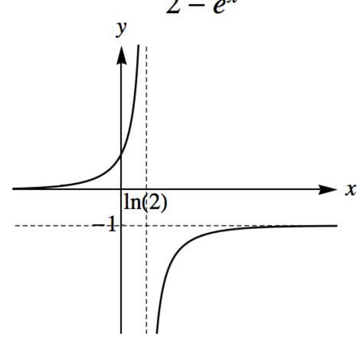
There are horizontal asymptotes at $\boxed{y=-1}$ and $\boxed{y=0}$.

Vertical asymptotes: Check $x = \ln 2$ where the denominator equals zero.

$$\lim_{x\to(\ln 2)^+} \frac{e^x}{2-e^x} \to \frac{2}{0^-} = -\infty \quad \text{or} \quad \lim_{x\to(\ln 2)^-} \frac{e^x}{2-e^x} \to \frac{2}{0^+} = \infty$$

There is a vertical asymptote at $\boxed{x = \ln 2}$.

$$y = \frac{e^x}{2-e^x}$$



(b) (13 pts)

(i) Domain: $\boxed{(-\infty, 5]}$.

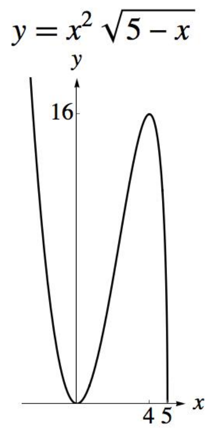
(ii)

$$y = x^2\sqrt{5-x} \Rightarrow y' = \frac{-x^2}{2\sqrt{5-x}} + 2x\sqrt{5-x}$$

Solve $y' = 0$.

$$y' = \frac{-x^2 + 4x(5-x)}{2\sqrt{5-x}} = \frac{-5x^2+20x}{2\sqrt{5-x}} = \frac{5x(-x+4)}{2\sqrt{5-x}} = 0 \Rightarrow x = 0, 4$$

Since $y'(-1) < 0, y'(1) > 0$, and $y'(9/2) < 0$, y is decreasing on $\boxed{(-\infty, 0), (4, 5)}$ and increasing on $\boxed{(0, 4)}$.



7. (20 pts) The following problems are not related.

(a) Find the sum of $\sum_{n=1}^{10} (\ln(3n) - \ln(2n))$. Simplify your answer.

(b) Let $g(x) = \int_0^x \tanh(t^2 - t - 2)\,dt$. Find $g'(x)$ and $g''(x)$.

(c) A sample of radioactive cesium-137 with an initial mass m of 50 mg decays at the rate of

$$\frac{dm}{dt} = -\frac{(\ln 2)m}{30} \text{ mg/year.}$$

Find an expression for $m(t)$, the mass remaining after t years. Simplify your answer.

Solution:

(a) (6 pts) $\sum_{n=1}^{10} (\ln(3n) - \ln(2n)) = \sum_{n=1}^{10} \ln\frac{3}{2} = \boxed{10\ln\frac{3}{2}}$.

(b) (8 pts) $g'(x) = \boxed{\tanh\left(x^2 - x - 2\right)}$ by FTC-1.

$g''(x) = \boxed{(2x-1)\,\mathrm{sech}^2\left(x^2 - x - 2\right)}$

(c) (6 pts) $m(t) = 50e^{-\frac{\ln 2}{30}t} = 50\left(e^{\ln 2^{-\frac{1}{30}}}\right)^t = \boxed{50\left(2^{-\frac{t}{30}}\right)}$ or $25\left(2^{1-\frac{t}{30}}\right)$ mg.