1. (21 pts) Evaluate each of the following.

(a) $\displaystyle\int \frac{\sec^2(1/x^2)}{x^3}\,dx$ (b) $\displaystyle\int_0^2 \frac{t}{\sqrt{1+2t^2}}\,dt$ (c) $\displaystyle\int_2^{x^2} \frac{3}{2t+1}\,dt$

Solution:

(a) (7 pts) Let $u = 1/x^2, du = -2/x^3\,dx$.

$$\int \frac{\sec^2(1/x^2)}{x^3}\,dx = \int -\frac{1}{2}\sec^2 u\,du = -\frac{1}{2}\tan u + C = \boxed{-\frac{1}{2}\tan(1/x^2) + C}$$

(b) (7 pts) Let $u = 1 + 2t^2, du = 4t\,dt$.

$$\int_0^2 \frac{t}{\sqrt{1+2t^2}}\,dt = \frac{1}{4}\int_1^9 u^{-1/2}\,du = \frac{1}{4}\Big[2\sqrt{u}\Big]_1^9 = \frac{1}{2}(3-1) = \boxed{1}$$

(c) (7 pts) Let $u = 2t+1, du = 2\,dt$.

$$\int_2^{x^2} \frac{3}{2t+1}\,dt = \frac{3}{2}\int_5^{2x^2+1} \frac{du}{u} = \frac{3}{2}\big[\ln|u|\big]_5^{2x^2+1} = \boxed{\frac{3}{2}\left(\ln\left|2x^2+1\right| - \ln 5\right)} = \frac{3}{2}\ln\left|\frac{2x^2+1}{5}\right|$$

2. (15 pts) Let $f(x) = x^4$ and $b > 0$ be constant.

(a) Find the average value of $f(x)$ on the interval $[0, b]$.

(b) Using your answer to part (a), find the appropriate value of c from the Mean Value Theorem for Integrals.

(c) Find the value of $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2bi}{n}\right)^4\frac{b}{n}$.

Solution:

(a) (5 pts) $\displaystyle f_{ave} = \frac{1}{b}\int_0^b x^4\,dx = \frac{1}{b}\left[\frac{x^5}{5}\right]_0^b = \frac{1}{b}\cdot\frac{b^5}{5} = \boxed{\frac{b^4}{5}}$

(b) (5 pts) Since f is a continuous function, there is a c in $[0, b]$ such that $f(c) = f_{ave}$.

$$f(c) = c^4 = \frac{b^4}{5} \;\Rightarrow\; c = \boxed{\sqrt[4]{\frac{b^4}{5}}} = \frac{b}{\sqrt[4]{5}}.$$

(c) (5 pts) $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2bi}{n}\right)^4\frac{b}{n} = 16\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{bi}{n}\right)^4\frac{b}{n} = 16\int_0^b x^4\,dx = 16\left[\frac{x^5}{5}\right]_0^b = \boxed{\frac{16b^5}{5}}$ since $\displaystyle\int_0^b x^4\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{bi}{n}\right)^4\frac{b}{n}$ with $\Delta x = \dfrac{b}{n}$ and $x_i = \dfrac{bi}{n}$.

3. (20 pts) The velocity of a particle moving along a line is $v(t) = \sqrt{t}\cos(t),\ \ 0 \le t \le 2\pi$.

   (a) Approximate the displacement of the particle using M_2, the midpoint approximation with two equal subintervals.

   (b) Write (but do not evaluate) integral(s) to calculate the total distance traveled by the particle. Express your answer without using absolute value signs.

   (c) Let $g(x) = \displaystyle\int_0^x \sqrt{t}\cos t\,dt,\ \ 0 \le x \le 2\pi$.

      i. On what intervals (if any) is g decreasing?

      ii. Find the value of $g'(\pi)$.

Solution:

(a) (6 pts) Let $\Delta t = \pi$. Then

$$M_2 = \Delta t\,(v(\pi/2) + v(3\pi/2)) = \pi\left(\sqrt{\pi/2}\cos(\pi/2) + \sqrt{3\pi/2}\cos(3\pi/2)\right) = \pi(0+0) = \boxed{0}$$

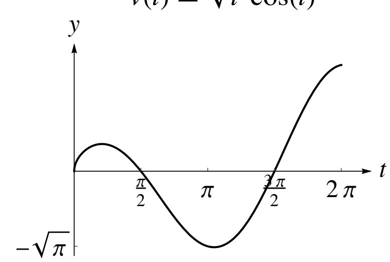
(b) (6 pts) Since $v(t) < 0$ where $\cos t < 0$ on $(\pi/2, 3\pi/2)$, the total distance is

$$\boxed{\int_0^{\pi/2} \sqrt{t}\cos t\,dt - \int_{\pi/2}^{3\pi/2} \sqrt{t}\cos t\,dt + \int_{3\pi/2}^{2\pi} \sqrt{t}\cos t\,dt}.$$

(c) i. (2 pts) Note that $g'(x) = \sqrt{x}\cos x$ by FTC-1. g is decreasing where $g'(x) = \sqrt{x}\cos x < 0$ on $\boxed{(\pi/2, 3\pi/2)}$.

   ii. (6 pts) $g'(\pi) = \sqrt{x}\cos x\Big|_{x=\pi} = \sqrt{\pi}\cos(\pi) = \boxed{-\sqrt{\pi}}$.

$v(t) = \sqrt{t}\ \cos(t)$



4. (20 pts) The following problems are not related.

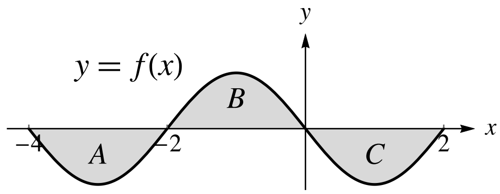
   (a) Evaluate $\displaystyle\sum_{k=10}^{100} k + \sum_{k=101}^{150} 2$.

   (b) Express the following in sigma notation and find the sum.

$$\ln(1/2) + \ln(2/3) + \ln(3/4) + \cdots + \ln(99/100)$$

   (c) Each of the regions A, B, and C bounded by the graph of f and the x-axis has an area of 7.

   Find the value of $\displaystyle\int_{-4}^{2} (f(x) + 2x + 3)\,dx$.



Solution:

(a) (6 pts) $\sum_{k=10}^{100} k + \sum_{k=101}^{150} 2 = \sum_{k=1}^{100} k - \sum_{k=1}^{9} k + 2(50) = \frac{100(101)}{2} - \frac{9(10)}{2} + 100 = 5050 - 45 + 100 = \boxed{5105}$.

(b) (6 pts)

$$\boxed{\sum_{k=1}^{99} \ln\left(\frac{k}{k+1}\right)} = \ln\frac{1}{2} + \ln\frac{2}{3} + \ln\frac{3}{4} + \cdots + \ln\frac{99}{100}$$

$$= \ln\left(\frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{4}} \cdots \frac{\cancel{99}}{100}\right) = \ln\left(\frac{1}{100}\right) = \boxed{-\ln(100)}$$

Alternate Solution:

$$\sum_{k=1}^{99} \ln\left(\frac{k}{k+1}\right) = (\ln 1 - \cancel{\ln 2}) + (\cancel{\ln 2} - \cancel{\ln 3}) + \cdots + (\cancel{\ln 99} - \ln 100)$$

$$= \ln 1 - \ln 100 = -\ln 100$$

(c) (8 pts)

$$\int_{-4}^{2} (f(x) + 2x + 3)\, dx = \int_{-4}^{2} f(x)\, dx + \int_{-4}^{2} (2x+3)\, dx$$

$$= -7 + \left[x^2 + 3x\right]_{-4}^{2} = -7 + ((4+6) - (16-12)) = -7 + 6 = \boxed{-1}$$

5. (24 pts) The following problems are not related.

(a) The function $h(x) = \dfrac{3}{2x+1}$ is one-to-one. Find h^{-1} and its range in interval notation.

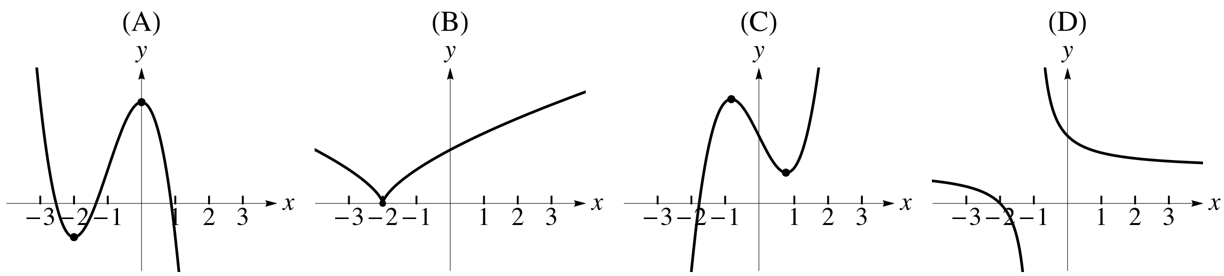
(b) Suppose that g and G are functions for which $G'(x) = g(x)$ and g is continuous. A table of values for G is shown below. Evaluate $\displaystyle\int_4^9 \frac{g(\sqrt{x})}{\sqrt{x}}\, dx$ and simplify your answer fully.

x	0	1	2	3	4	5	9
$G(x)$	1	3	9	16	32	44	64

(c) Suppose Newton's Method is used to find the root(s) of a function $f(x)$ given an initial approximation of $x_1 = 0$. The next two approximations have the values $x_2 = 1$ and $x_3 = 0$.

i. The graph of $y = f(x)$ is shown below. Write the letter of the graph that matches f. No justification is necessary.

ii. Will the approximations $x_4, x_5, \ldots$, converge toward a root of f? Write Yes, No, or Maybe and explain your answer.



Solution:

(a) (8 pts) $y = \dfrac{3}{2x+1} \Rightarrow 2xy + y = 3 \Rightarrow x = \dfrac{3-y}{2y} \Rightarrow h^{-1}(x) = \boxed{\dfrac{3-x}{2x}}$.

The range of h^{-1} equals the domain of h which is $\boxed{(-\infty, -1/2) \cup (-1/2, \infty)}$.

(b) (8 pts) Let $u = \sqrt{x}$, $du = dx/(2\sqrt{x})$.

$$\int_4^9 \frac{g(\sqrt{x})}{\sqrt{x}}\, dx = \int_2^3 2g(u)\, du = 2G(u)\Big|_2^3 = 2(G(3) - G(2)) = 2(16 - 9) = \boxed{14}$$

(c) i. (4 pts) $\boxed{\text{C}}$

ii. (4 pts) $\boxed{\text{No}}$ because the approximations will cycle between 0 and 1.