1. (28 pts, 7 pts each) The following four problems are not related. Be sure to simplify your answers.

(a) Let $f(x) = \tan^2(\pi x)$. Find $f'(1/4)$

Solution:

(5 pts) $f'(x) = 2\pi \tan(\pi x) \sec^2(\pi x)$

(2 pts) $f'(1/4) = 2\pi \tan(\pi/4) \sec^2(\pi/4) = 2\pi(1)\left(\sqrt{2}\right)^2 = \boxed{4\pi}$

(b) Find the derivative of $g(x) = \dfrac{x^2}{\sqrt{3x^2+4}}$ at $x = 2$.

Solution:

(5 pts) $$g'(x) = \frac{\sqrt{3x^2+4}(2x) - x^2 \cdot \dfrac{6x}{2\sqrt{3x^2+4}}}{3x^2+4} = \frac{2x\sqrt{3x^2+4} - \dfrac{3x^3}{\sqrt{3x^2+4}}}{3x^2+4}$$

(2 pts) $$g'(2) = \frac{4(4) - \dfrac{24}{4}}{16} = \boxed{\frac{5}{8}}$$

(c) Consider the curve $Ax^2 + By^2 = 40$.

i. Use implicit differentiation to find dy/dx.

ii. Suppose the curve has the tangent line $y = 2 + \frac{2}{9}(x+1)$ at the point $(-1, 2)$. Find the constants A and B.

Solution:

i. (3 pts) $2Ax + 2By \cdot \dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = \boxed{\dfrac{-Ax}{By}}$

ii. (4 pts) The point $(-1, 2)$ lies on the curve so $A + 4B = 40$.

The tangent slope at $(-1, 2)$ is $y' = \dfrac{A}{2B} = \dfrac{2}{9}$.

Combining the two equations we find that $\boxed{A = 4, B = 9}$.

(d) Find the derivative of $y = x^2|x-1|$. Express your answer as a piecewise defined function without absolute value signs.

Solution: (7pts)

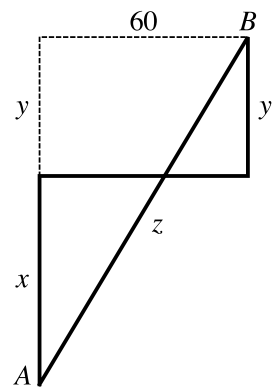
$$y = x^2|x-1| = \begin{cases} x^3 - x^2, & x \geq 1 \\ -x^3 + x^2, & x < 1 \end{cases}$$

$$y' = \boxed{\begin{cases} 3x^2 - 2x, & x > 1 \\ -3x^2 + 2x, & x < 1 \end{cases}}$$

Note that $y'(1)$ is undefined because y' approaches 1 from the right and -1 from the left.

2. (12 pts) At noon, ship A is 60 km west of ship B. Ship A is sailing south at 15 km/h and ship B is sailing north at 5 km/h. How fast is the distance between the ships changing at 4:00 PM?

Solution:



Let x represent the distance that ship A travels, y the distance that ship B travels, and z the distance between the two ships.

We are given that $dx/dt = 15$ km/h and $dy/dt = 5$ km/h. We wish to find dz/dt at 4 p.m. when $x = 60$, $y = 20$, and $z = \sqrt{60^2 + 80^2} = 100$ km.

$$\begin{aligned} z^2 &= (x+y)^2 + 60^2 \\ \cancel{2}z\frac{dz}{dt} &= \cancel{2}(x+y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right) \\ 100\frac{dz}{dt} &= (60+20)(15+5) = 1600 \\ \frac{dz}{dt} &= \boxed{16 \text{ km/h}} \end{aligned}$$

Alternate Solution:

Let $D = x + y$. Then $dD/dt = dx/dt + dy/dt = 20$ and $D = 60 + 20 = 80$ at 4 p.m.

$$z^2 = D^2 + 60^2 \Rightarrow \cancel{2}z\frac{dz}{dt} = \cancel{2}D \cdot \frac{dD}{dt} \Rightarrow 100\frac{dz}{dt} = 80(20) \Rightarrow \frac{dz}{dt} = 16 \text{ km/h}$$

3. (24 pts) Consider the function $y = x\sqrt{6-x}$.

   (a) Find the domain of the function. Express your answer in interval notation.

   (b) Find the x- and y-intercepts of the function.

   (c) The first derivative is $y' = \dfrac{12-3x}{2\sqrt{6-x}}$. On what intervals is y increasing? decreasing?

   (d) Find x and y coordinates of the local maximum and minimum extrema, if any.

   (e) The second derivative is $y'' = \dfrac{3x-24}{4(6-x)^{3/2}}$. On what intervals is y concave up? concave down?

(f) Find the x and y coordinates of the inflection points, if any.

(g) Sketch a graph of y. Clearly label any intercepts, local extrema, and inflection points.

Solution:

(a) (3 pts) The domain is $\boxed{(-\infty, 6]}$.

(b) (3 pts) The intercepts are $\boxed{(0,0)}$ and $\boxed{(6,0)}$.

(c) (4 pts) The first derivative $y' = 0$ at $x = 4$ and y' is undefined at the endpoint $x = 6$.

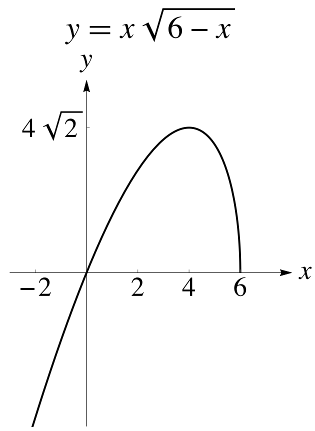
Intervals	y'	y
$x < 4$	+	increasing on $(-\infty, 4)$
$4 < x < 6$	–	decreasing on $(4, 6)$

(d) (3 pts) There is a local maximum value at $\boxed{\left(4, 4\sqrt{2}\right)}$.

(e) (4 pts) The second derivative y'' does not equal 0 and y'' is undefined at the endpoint $x = 6$. The curve is $\boxed{\text{concave down on } (-\infty, 6)}$.

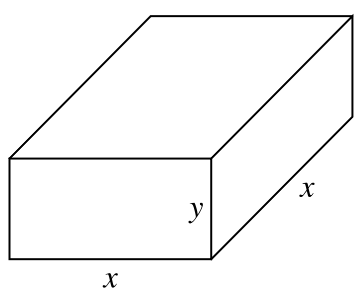
(f) (3 pts) There are $\boxed{\text{no inflection points}}$.

(g) (4 pts)



4. (12 pts) If 1200 cm^2 of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

Solution:



Let x be the length of the base and y be the height of the box. The surface area is

$1200 = x^2 + 4xy \;\Rightarrow\; y = \dfrac{1200 - x^2}{4x} = \dfrac{300}{x} - \dfrac{x}{4}$. We wish to maximize the volume V.

$$V = x^2 y = x^2\left(\frac{300}{x} - \frac{x}{4}\right) = 300x - \frac{x^3}{4}.$$

$$V' = 300 - \frac{3}{4}x^2$$

$$V' = 0 \;\Rightarrow\; 300 = \frac{3}{4}x^2 \;\Rightarrow\; x^2 = 400 \;\Rightarrow\; x = 20 \text{ since } x > 0$$

$$V'' = -\frac{3}{2}x$$

Since $V'' < 0$ for all $x > 0$, the curve is concave down and has an absolute maximum when $x = 20$, $y = 10$. The largest possible volume is therefore $V = \boxed{4000 \text{ cm}^3}$.

5. (24 pts, 6pts each) The following four problems are not related and do not require justification.

(a) The linearization function for $f(x) = (1+x)^k$ at $a = 0$ is $L(x) = 1 + kx$. Suppose $L(x)$ is used to estimate the value of $\sqrt[3]{0.94^2}$.

   i. What values of k and x should be used?

   ii. What is the estimated value? Simplify your answer.

   Solution:

   i. (4 pts) $k = \boxed{2/3}$, $x = \boxed{-0.06}$.

   ii. (2 pts) $L(-0.06) = 1 + \frac{2}{3}(-0.06) = \boxed{0.96}$

(b) Consider the function $y = \dfrac{6x^2 - 15x - 2}{2x - 3}$.

   i. Find the slant asymptote(s) of y or write "None" if none exists.

   ii. Write down (but do not evaluate) the appropriate limit(s) to confirm your answer for part (i).

   Solution:

   i. (4 pts)

$$\begin{array}{r|rrr} & 3x & -3 & \\ \hline 2x-3 & 6x^2 - 15x & -2 & \\ & -6x^2 + 9x & & \\ \hline & -6x & -2 & \\ & 6x & -9 & \\ \hline & & -11 & \end{array}$$

   There is a slant asymptote at $\boxed{y = 3x - 3}$.

ii. (2 pts) $\lim_{x\to\pm\infty}\left[\left(3x-3-\frac{11}{2x-3}\right)-(3x-3)\right]$ or $\lim_{x\to\pm\infty}\left[\frac{6x^2-15x-2}{2x-3}-(3x-3)\right]$.

(c) Suppose $g(x)$ is differentiable for all x. Which of the following statements A1 to A4 *must* be true? Write down the numbers of all statements that are necessarily true. If none of the statements are true, write "None".

A1: g is continuous on the interval $[-3, 10]$.

A2: g has absolute maximum and minimum values on the interval $[-3, 10]$.

A3: $g(c) = 0$ for some value of c.

A4: $g'(c) = 0$ for a number c in $(-3, 10)$.

Solution:

$\boxed{A1.}$ Since g is differentiable for all x, it is continuous for all x.

$\boxed{A2.}$ By the Extreme Value Theorem.

(d) Suppose $-2 \le h'(x) \le 2$ for all x. Which of the following statements B1 to B4 *must* be true? Write down the numbers of all statements that are necessarily true. If none of the statements are true, write "None".

B1: $h(c)$ exists for all values of c.

B2: $h(c) = 0$ for some value of c.

B3: $h'(c) = 0$ for a number c in $(-2, 2)$.

B4: $h'(c) = \frac{h(2)-h(-2)}{4}$ for a number c in $(-2, 2)$.

Solution:

$\boxed{B1.}$ If h is differentiable at c, then it is continuous at c and therefore $h(c)$ exists.

$\boxed{B4.}$ Since h is continuous and differentiable for all x, it satisfies the conclusion of the Mean Value Theorem on any closed interval.