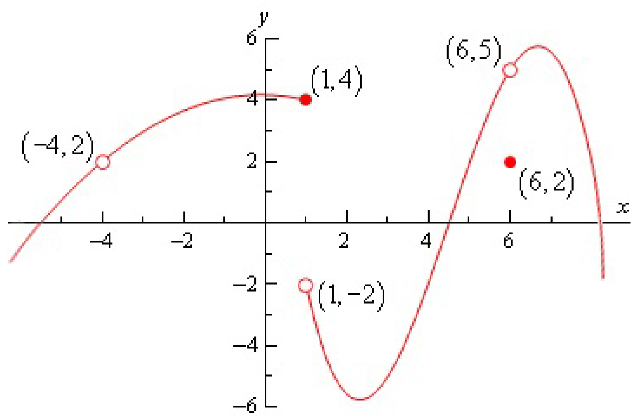
1. (16 points) Use the graph of f, shown below, to answer the following questions:

   (a) What is $\lim_{x\to 1^+} f(x)$?

   (b) What is $\lim_{x\to 6} f(x)$?

   (c) State the definition of continuity of a function f at a number a.

   (d) At what values of x does the function shown in the graph below fail to be continuous? Explain how each x-value fails to satisfy the definition of continuity. Be specific and provide details in your answer.



Solution:

(a) (2 pts) $\boxed{-2}$

(b) (2 pts) $\boxed{5}$

(c) (3 pts) $f(a) = \lim_{x\to a} f(x)$

Also acceptable: $f(a) = \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$
or
3-part definition: $f(a)$ exists, $\lim_{x\to a} f(x)$ exists, and $f(a) = \lim_{x\to a} f(x)$.

(d) (9 pts) f is discontinuous at $x = \boxed{-4, 1, 6}$.
At $x = -4$, $f(-4)$ does not exist.
At $x = 1$, $\lim_{x\to 1} f(x)$ does not exist because the limit from the left is 4 and the limit from the right is -2.
At $x = 6$, $f(6) = 2$ and $\lim_{x\to 6} f(x) = 5$ so $f(6) \neq \lim_{x\to 6} f(x)$.

2. (36 points) The following problems are not related.

   (a) Find $\lim_{x\to 0} 2x\csc(5x)$.

   (b) Find $\lim_{t\to -1} \dfrac{|t+1|}{t^2-1}$.

   (c) Consider $\lim_{h\to 0} \dfrac{\sqrt{9+h}-3}{h}$.

      i. Evaluate the limit.

      ii. If the limit corresponds to the slope of the tangent line to a curve $y = f(x)$ at a point $(a, f(a))$, what is $f(x)$ and what is a?

   (d) Let $g(x) = x^3 - \cos(x) - 1$.

      i. Show that g has a positive root. Justify your answer.

      ii. Find the instantaneous rate of change of g at $x = 4\pi/3$.

Solution:

(a) (7 pts) $\lim_{x\to 0} 2x\csc(5x) = \lim_{x\to 0} 2\cdot\dfrac{x}{\sin(5x)} = \lim_{x\to 0} 2\cdot\dfrac{1}{5}\cdot\dfrac{5x}{\sin(5x)} = \boxed{\dfrac{2}{5}}$

(b) (7 pts)

From the right:

$$\lim_{t\to -1^+} \frac{|t+1|}{t^2-1} = \lim_{t\to -1^+} \frac{t+1}{(t+1)(t-1)} = \lim_{t\to -1^+} \frac{1}{t-1} = -\frac{1}{2}$$

From the left:

$$\lim_{t\to -1^-} \frac{|t+1|}{t^2-1} = \lim_{t\to -1^-} \frac{-(t+1)}{(t+1)(t-1)} = \lim_{t\to -1^-} \frac{-1}{t-1} = \frac{1}{2}$$

Therefore $\lim_{t\to -1} \dfrac{|t+1|}{t^2-1}$ $\boxed{\text{does not exist}}$.

(c) i. (7 pts)

$$\lim_{h\to 0} \frac{\sqrt{9+h}-3}{h}\cdot\frac{\sqrt{9+h}+3}{\sqrt{9+h}+3} = \lim_{h\to 0} \frac{\cancel{9}+h-\cancel{9}}{h\left(\sqrt{9+h}+3\right)} = \lim_{h\to 0} \frac{1}{\sqrt{9+h}+3} = \boxed{\frac{1}{6}}$$

ii. (4 pts) $\boxed{f(x) = \sqrt{x}}$, $\boxed{a = 9}$

(d) i. (7 pts) By the Intermediate Value Theorem, since $g(x) = x^3 - \cos(x) - 1$ is continuous and

$$g(0) = -2 < 0, \quad g(\pi) = \pi^3 > 0,$$

g has a positive root in $(0, \pi)$.

ii. (4 pts) $g'(x) = 3x^2 + \sin x, \quad g'(4\pi/3) = \boxed{\dfrac{16\pi^2}{3} - \dfrac{\sqrt{3}}{2}}$.

3. (18 points) Let $h(x) = \dfrac{x^2+5x+6}{2x^2+5x+2}$.

   (a) What is the domain of h? Express your answer in interval notation.

   (b) Find all horizontal asymptotes of h. Justify your answer using limits.

   (c) Find all vertical asymptotes of h. Justify your answer using limits.

Solution:

(a) (4 pts) $h(x) = \dfrac{(x+2)(x+3)}{(2x+1)(x+2)}$. The domain is $\boxed{(-\infty,-2)\cup(-2,-1/2)\cup(-1/2,\infty)}$.

(b) (7 pts) $\displaystyle\lim_{x\to\infty} \frac{x^2+5x+6}{2x^2+5x+2}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty}\frac{1+\frac{5}{x}+\frac{6}{x^2}}{2+\frac{5}{x}+\frac{2}{x^2}} = \frac{1}{2}$

$\displaystyle\lim_{x\to-\infty}\frac{1+\frac{5}{x}+\frac{6}{x^2}}{2+\frac{5}{x}+\frac{2}{x^2}} = \frac{1}{2}$

There is a horizontal asymptote at $\boxed{y=1/2}$.

(c) (7 pts) h simplifies to $\dfrac{x+3}{2x+1}$. Check $x=-1/2$ where the denominator equals 0.

$\displaystyle\lim_{x\to-1/2^+}\frac{x+3}{2x+1} \to \frac{5/2}{0^+} = \infty$

There is a vertical asymptote at $\boxed{x=-1/2}$.

(Note: There is a removable discontinuity at $x=-2$.)

4. (18 points) Let $f(x) = 3x^2-5$.

   (a) Use the limit definition of the derivative to calculate $f'(x)$.

   (b) Find an equation for the line tangent to $y=f(x)$ at $x=-1$.

   (c) Sketch a graph to illustrate the precise definition of $\displaystyle\lim_{x\to-1}\left(3x^2-5\right)$. Clearly label δ and ϵ in your graph.

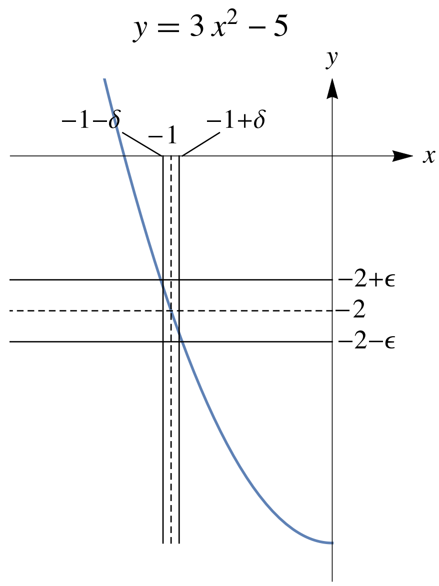
Solution:

(a) (8 pts)

$$\begin{aligned} f'(x) &= \lim_{h\to0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to0}\frac{3(x+h)^2-5-\left(3x^2-5\right)}{h} \\ &= \lim_{h\to0}\frac{3\left(x^2+2xh+h^2\right)-\cancel{5}-3x^2+\cancel{5}}{h} \\ &= \lim_{h\to0}\frac{\cancel{3x^2}+6xh+3h^2-\cancel{3x^2}}{h} \\ &= \lim_{h\to0}(6x+3h) = \boxed{6x} \end{aligned}$$

(b) (4 pts) $y(-1)=-2$ and $y'(-1)=-6$. The tangent line is $\boxed{y+2=-6(x+1)}$ or $y=-6x-8$.

(c) (6 pts)



5. (12 points) Sketch the graph of a single function g that satisfies all of the following conditions. No explanation is necessary.

g is odd	$\lim_{x\to 2} g(x) = \infty$
g is defined for all $x \neq \pm 2$	$\lim_{x\to -\infty} g(x) = 0$
$\lim_{x\to 4} g(x) = -6$	$g(4) \neq \lim_{x\to 4} g(x)$

Solution: Here is one possible solution.