1. (28 points) Evaluate the following.

(a) \[ \int \frac{(7 - \sqrt{t})^3}{\sqrt{t}} \, dt \]

(b) \[ \int_0^{\pi/4} \frac{\sec^2 \theta}{(1 + 7 \tan \theta)^{2/3}} \, d\theta \]

(c) The area of the shaded region shown below.

Solution:

(a) Let \( u = 7 - \sqrt{t} \), \( du = -1/(2\sqrt{t}) \, dt \). Then \( -2 \, du = dt/\sqrt{t} \).

\[
\int \frac{(7 - \sqrt{t})^3}{\sqrt{t}} \, dt = -2 \int u^3 \, du = -2 \cdot \frac{u^4}{4} + C = \frac{1}{2} (7 - \sqrt{t})^4 + C
\]

(b) Let \( u = 1 + 7 \tan \theta \), \( du = 7 \sec^2 \theta \, d\theta \). Then \( du/7 = \sec^2 \theta \, d\theta \). The \( u \)-limits are \( u(0) = 1 \) and \( u(\pi/4) = 8 \).

\[
\int_0^{\pi/4} \frac{\sec^2 \theta}{(1 + 7 \tan \theta)^{2/3}} \, d\theta = \frac{1}{7} \int_1^8 u^{-2/3} \, du = \frac{1}{7} \cdot 3u^{1/3} \bigg|_1^8 = \frac{3}{7} (\sqrt[3]{8} - \sqrt[3]{1}) = \frac{3}{7}
\]

(c) The area \( A \) of the shaded region equals the area of the enclosing rectangle minus the area \( I \) under the curve. First calculate the value of \( I \).

\[
I = \int_0^2 x\sqrt{4 - x^2} \, dx
\]

Let \( u = 4 - x^2 \), \( du = -2x \, dx \). Then \( -du/2 = x \, dx \). The \( u \)-limits are \( u(0) = 4 \) and \( u(2) = 0 \).

\[
I = -\frac{1}{2} \int_4^0 u^{1/2} \, du = -\frac{1}{2} \cdot \frac{2}{3} u^{3/2} \bigg|_4^0 = -\frac{1}{3} (0 - 4^{3/2}) = \frac{8}{3}
\]

The area of the shaded region equals the area of the enclosing rectangle minus \( I \).

\[
A = 3(2) - I = 6 - \frac{8}{3} = \frac{10}{3}
\]
2. (22 points) The following questions are related to the function \( y = x \sin x \), which is increasing on \([0, \pi/2]\).

(a) Verify by differentiation that the following formula is correct.

\[
\int x \sin x \, dx = -x \cos x + \sin x + C
\]

(b) Use the Min-Max Comparison Property to find lower and upper bounds \( L \) and \( U \) for the integral of \( x \sin x \) on \([0, \pi/2]\) such that

\[
L \leq \int_0^{\pi/2} x \sin x \, dx \leq U.
\]

(c) Use the formula in part (a) to find the exact value of \( \int_0^{\pi/2} x \sin x \, dx \).

(d) Suppose the velocity function (in m/s) for a particle moving along a line is \( v(t) = -t \sin t \), \( 0 \leq t \leq \pi/2 \).

i. Find the displacement of the particle.

ii. Find the total distance traveled by the particle.

(e) Use the Fundamental Theorem of Calculus to find the derivative of \( \int_0^{\sqrt{x}} t \sin t \, dt \).

Solution:

(a) Differentiate the function on the right side of the equation using the product rule. Verify that the derivative equals the integrand.

\[
\frac{d}{dx}(-x \cos x + \sin x) = \sin x - \cos x + \cos x = x \sin x
\]

(b) Since the function is increasing on \([0, \pi/2]\), the minimum value of \( y \) is \( m = y(0) = 0 \) and the maximum value is \( M = y(\pi/2) = \pi/2 \). By the Min-Max Comparison Property,

\[
m(b - a) \leq \int_0^{\pi/2} x \sin x \, dx \leq M(b - a)
\]

\[
0 \left( \frac{\pi}{2} \right) \leq \int_0^{\pi/2} x \sin x \, dx \leq \left( \frac{\pi}{2} \right) \left( \frac{\pi}{2} \right)
\]

\[
0 \leq \int_0^{\pi/2} x \sin x \, dx \leq \frac{\pi^2}{4}
\]

The lower bound is \( L = 0 \) and the upper bound is \( U = \pi^2/4 \).

(c) \( \int_0^{\pi/2} x \sin x \, dx = \left[-x \cos x + \sin x\right]_0^{\pi/2} = (0 + 1) - (0 + 0) = 1 \)

(d) Use the result from part (c).

i. The displacement equals

\[
\int_0^{\pi/2} v(t) \, dt = \int_0^{\pi/2} -t \sin t \, dt = -\int_0^{\pi/2} x \sin x \, dx = -1
\]
II. Since $v(0) = 0$ and $v(t)$ is a decreasing function, then $|v(t)| = -v(t)$. The total distance traveled is
\[
\int_0^{\pi/2} |v(t)| \, dt = \int_0^{\pi/2} -v(t) \, dt = \int_0^{\pi/2} t \sin t \, dt = 1.
\]
(e) Apply FTC and the chain rule.
\[
d \frac{d}{dx} \int_0^{\sqrt{x}} t \sin t \, dt = (\sqrt{x} \sin \sqrt{x}) \left( \frac{1}{2\sqrt{x}} \right) = \frac{1}{2} \sin \sqrt{x}
\]
For problems 3-5 refer to the graph of the continuous function $g$ shown below, defined on the interval $[0, 7]$. The graph of $g$ consists of a quarter-circle and three straight lines.

3. (28 points)

(a) For each of the following intervals, find a definite integral (or integrals) of the form $\int_a^b f(x) \, dx$ to represent the area of the region between $g$ and the $x$-axis. (For example, the area of the region on $[6, 7]$ can be represented by the integral $\int_6^7 2 \, dx$.)

i. $[0, 2]$  
ii. $[4, 6]$  

(b) Find the value of $\int_0^7 g(x) \, dx$.

(c) Find the value of $\int_0^7 |g(x)| \, dx$.

(d) Determine the average value of $g$.

(e) Use the graph of $g$ to estimate the $x$-value(s) that satisfy the Mean Value Theorem for Integrals. Round your answer to the nearest integer.

**Solution:**

(a) 

i. $\int_0^2 \sqrt{4 - x^2} \, dx$

ii. Here are three possible solutions: $\int_4^6 |-2x + 10| \, dx$ or $\int_4^5 2(-2x + 10) \, dx$

or $\int_4^5 (-2x + 10) \, dx - \int_5^6 (-2x + 10) \, dx$
(b) The integral equals the negative of the area of a quarter-circle on $[0, 2]$ plus the area of a triangle on $[2, 5]$ plus the negative of the area of a trapezoid on $[5, 7]$.

$$
\int_0^7 g(x) \, dx = -\frac{1}{4} \left( \pi \left( 2^2 \right) \right) + \frac{1}{2} (3)(2) - \frac{1}{2} (2)(2 + 1) = -\pi + 3 - 3 = -\pi
$$

(c) Add the areas found in part (b).

$$
\int_0^7 |g(x)| \, dx = \pi + 3 + 3 = \pi + 6
$$

(d) Use the result from part (b).

$$
g_{ave} = \frac{1}{b - a} \int_0^7 g(x) \, dx = \frac{1}{7 - 0} (-\pi) = -\frac{\pi}{7}
$$

(e) Since $g_{ave}$ is $-\pi/7$, slightly greater than $-1/2$, there are two values of $x$ that equal $g_{ave}$, at approximately $x = \frac{2}{7}$ and $x = \frac{5}{7}$ as shown in the solution graph above.

4. (10 points) Let $h(x) = \int_0^x g(t) \, dt$, $0 \leq x \leq 7$, where $g$ is the function shown in the graph above.

(a) On what interval(s) is $h$ decreasing?

(b) At what values of $x$ does $h$ attain absolute maximum and minimum values?

(c) What is the value of $h'(3)$?

Solution:

(a) $h$ is decreasing when $h'(x) = f(x) < 0$ on $[0, 2]$ and $[5, 7]$.

(b) $h$ has an absolute maximum value of 0 at $x = 0$ and an absolute minimum value of $-\pi$ at $x = \frac{2}{7}$ and $x = \frac{5}{7}$. Note that since the area of the triangle above the $x$-axis equals the area of the trapezoid below the $x$-axis, the absolute minimum is reached twice.

(c) By the Fundamental Theorem of Calculus, $h'(x) = g(x)$ so $h'(3) = g(3) = 1$ from the graph.

5. (12 points) Suppose we wish to find the inverse of the function $g$ shown above, which is not one-to-one.

If we restrict the domain of $g$ to $[0, b]$ where the function is one-to-one, we can find the inverse function $g^{-1}$ for the restricted $g$ function.

(a) What is the largest possible value of $b$?

(b) What is the domain of $g^{-1}$ if the domain of $g$ is $[0, b]$?

(c) Copy the graph of $g$ on $[0, b]$. Add the graph of $g^{-1}$.

(d) Find the values of the following:

i. $(g^{-1})'(0)$

ii. $(g^{-1})'(1)$

Solution:

(a) The largest possible value of \( b \) is 4. If \( b > 4 \) then there exist values \( x_1 \) and \( x_2 \) such that \( g(x_1) = g(x_2) \).

(b) The domain of \( g^{-1} \) equals the range of \( g \) on \([0, b]\): \([-2, 2]\).

(c) 

(d) i. The slope of \( g^{-1} \) at \((0, 2)\) is the reciprocal of the slope of \( g \) at \((2, 0)\). Since \( g'(2) \) is undefined, so is \((g^{-1})'(0)\), as seen in the graph.

ii. The slope of \( g^{-1} \) at \((1, 3)\) is the reciprocal of the slope of \( g \) at \((3, 1)\). Since \( g'(3) = 1 \), \((g^{-1})'(1) = 1\), as seen in the graph.