1. [14 pts] Solve the following equations.
   (a) \(2^{x^2-6x} = \frac{1}{32}\)  
   (b) \(\log(x + 2) + \log(x - 1) = 1\)

**Solution:**
(a)
\[2^{x^2-6x} = 2^{-5} \iff x^2 - 6x = 5 \iff x^2 - 6x + 5 = 0 \iff (x - 5)(x - 1) = 0 \iff x = 1 \text{ or } x = 5\]
Alternatively,
\[
\log_2(2^{x^2-6x}) = \log_2 \frac{1}{32} = \log_2 2^{-5}
\]
\[(x^2 - 6x) \log_2 2 = -5 \log_2 2
\]
\[x^2 - 6x = -5
\]
\[x^2 - 6x + 5 = 0
\]
\[(x - 5)(x - 1) = 0
\]
\[x = 1 \text{ or } x = 5\]

(b)
\[\log(x + 2) + \log(x - 1) = 1
\]
\[\log((x + 2)(x - 1)) = 1
\]
\[(x + 2)(x - 1) = 10^1 \text{ or } 10^{\log((x+2)(x-1))} = 10^1
\]
\[x^2 + x - 2 = 10
\]
\[x^2 + x - 12 = 0
\]
\[(x + 4)(x - 3) = 0
\]
\[x = -4 \text{ or } x = 3\]
Checking these potential solutions in the original equation shows that \(x = -4\) is not a solution since we cannot take logarithms of negative numbers. Thus the solution is \(x = 3\).

2. (a) [7 pts] Using any method you choose, show that the function \(f(x) = \frac{2x}{x + 1}\) is one-to-one and find its inverse.
(b) [7 pts] Consider the function \(f(x) = \sqrt{x^4 + x^3 + 1}\) with \(x \geq 0\). Evaluate the derivative of \(f^{-1}(x)\) at \(\sqrt{3}\), that is, find \((f^{-1})'(\sqrt{3})\). You can assume that \(f\), with the restricted domain, is one-to-one.

**Solution:**
(a) Using a graph: Graph of \(f(x)\) passes the horizontal line test and is thus one-to-one.
Using algebra: Let $x_1$ and $x_2$ be two points in the domain of $f(x)$ and assume that $f(x_1) = f(x_2)$. Then

\[
\frac{2x_1}{x_1 + 1} = \frac{2x_2}{x_2 + 1}
\]

\[
\iff 2x_1(x_2 + 1) = 2x_2(x_1 + 1)
\]

\[
\iff 2x_1x_2 + 2x_1 = 2x_2x_1 + 2x_2
\]

\[
\iff 2x_1 = 2x_2
\]

\[
\iff x_1 = x_2
\]

Since $f(x_1) = f(x_2) \implies x_1 = x_2$, $f(x)$ is one-to-one.

Using calculus: $f'(x) = \frac{(x + 1)(2) - 2x(1)}{(x + 1)^2} = \frac{2}{(x + 1)^2} > 0$ for all $x$ in the domain of $f$, which is $(-\infty, -1) \cup (-1, \infty)$, implying that $f(x)$ is increasing throughout its domain and therefore one-to-one.

To find the inverse, let $y = \frac{2x}{x + 1}$. Then

\[
x = \frac{2y}{y + 1}
\]

\[
x(y + 1) = 2y
\]

\[
xy - 2y = -x
\]

\[
y(x - 2) = -x
\]

\[
y = \frac{-x}{x - 2}
\]

\[
f^{-1}(x) = \frac{x}{2 - x}
\]

(b) Begin by finding $f^{-1}(\sqrt{3})$. This is accomplished by solving $\sqrt{3} = \sqrt{x^4 + x^3 + 1}$ which, by inspection, has solution $x = 1$. Thus $f^{-1}(\sqrt{3}) = 1$. Next we find $f'(x) = \frac{4x^3 + 3x^2}{2\sqrt{x^4 + x^3 + 1}}$. Finally, then

\[
(f^{-1})'((\sqrt{3})) = \frac{\frac{1}{f'(f^{-1}((\sqrt{3}))}}{f'(1)} = \frac{1}{\frac{4(1)^2 + 3(1)^2}{2\sqrt{1^4 + 1^3 + 1}}} = \frac{1}{\frac{2}{\sqrt{3}}} = \frac{2\sqrt{3}}{7}
\]
3. The following questions are unrelated.

(a) [7 pts] Differentiate \( g(w) = \frac{\ln w}{1 + \ln 2w} \), simplifying your answer completely.

(b) [7 pts] Use logarithmic differentiation to find \( y' \) if \( y = \sqrt{x+1} \). Simplify your answer completely.

(c) [7 pts] Consider the function \( f(x) = 2 \sin^{-1}(\ln x) \).
   i. Find the domain of \( f \).
   ii. Find the range of \( f \).
   iii. Find the derivative of \( f \).

(d) [7 pts] Find the intervals of increase or decrease, the intervals of concavity, and the points of inflection of the function \( f(x) = (1-x)e^{-x} \).

**SOLUTION:**

(a)

\[
g'(w) = \frac{(1 + \ln 2w) \left( \frac{1}{w} \right) - \ln w \left( \frac{1}{w} \right)(2)}{(1 + \ln 2w)^2} = \frac{1 + \ln 2}{w(1 + \ln 2w)^2}
\]

(b) Begin by taking the natural logarithm of both sides of the equation.

\[
\ln y = \ln \sqrt{x+1} = \ln[x(x+1)]^{1/2} = \frac{1}{2} \ln [x + \ln(x+1)]
\]

Then differentiate and simplify:

\[
\frac{1}{y} \frac{dy}{dx} = \frac{1}{2} \left( \frac{1}{x} + \frac{1}{x+1} \right) = \frac{1}{2} \left( \frac{x + 1 + x}{x(x+1)} \right) = \frac{2x + 1}{2x(x+1)}
\]

\[
\Rightarrow \frac{dy}{dx} = \sqrt{x(x+1)} \left( \frac{2x + 1}{2(x+1)} \right) = \frac{2x + 1}{2\sqrt{x(x+1)}}
\]

(c) i. The domain of \( \sin^{-1} x \) is \([-1, 1]\). Thus we must have \(-1 \leq \ln x \leq 1 \implies e^{-1} \leq e^{\ln x} \leq e^1 \) or \( x \in [\frac{1}{e}, e] \).

   ii. The range of \( \sin^{-1} x \) is \([-\pi, \pi]\). Thus the range of \( f \) is \([\frac{-\pi}{2}, \frac{\pi}{2}]\) (vertical stretching).

   iii. Using the chain rule yields \( f'(x) = \frac{2}{x \sqrt{1 - (\ln x)^2}} \)

(d)

\[
f'(x) = (1-x)(e^{-x})(-1) + e^{-x}(-1) = -e^{-x} + xe^{-x} - e^{-x} = -2e^{-x} + xe^{-x} = e^{-x}(x - 2)
\]

\[
f'(x) \quad \frac{2}{x} \quad + \quad x
\]

From the above chart, \( f(x) \) is decreasing on \((-\infty, 2)\) and increasing on \((2, \infty)\).

\[
f''(x) = e^{-x}(1) + (x-2)e^{-x}(-1) = e^{-x} - xe^{-x} + 2e^{-x} = 3e^{-x} - xe^{-x} = e^{-x}(3-x)
\]

\[
f''(x) \quad + \quad 3 \quad - \quad x
\]

From the above chart, \( f(x) \) is concave up on \((-\infty, 3)\) and concave down on \((3, \infty)\). \( f \) has an inflection point at \((3, f(3)) = (3, (1 - 3)e^{-3}) = (3, -2/e^3)\)

\[\boxed{\text{Solution Complete}}\]
4. [24 pts] Evaluate the following integrals.

(a) \( \int \frac{e^{2/x}}{x^2} \, dx \)  
(b) \( \int \left( t^{-1} + \frac{1}{1 - 3t} \right) \, dt \)  
(c) \( \int_{-\frac{2\pi}{3}}^{\frac{2}{3}} \frac{1}{4t^2 + 9} \, dt \)

**SOLUTION:**

(a) Let \( u = 2/x \) so that \( du = -2/x^2 \, dx \) \( \Rightarrow \) \( dx = -(x^2/2) \, du \). Then

\[
\int \frac{e^{2/x}}{x^2} \, dx = \int e^u \left( -\frac{x^2}{2} \right) \, du = -\frac{1}{2} \int e^u \, du = -\frac{1}{2} e^u + C = -\frac{1}{2} e^{2/x} + C
\]

(b) We can write the original integral as \( \int \left( t^{-1} + \frac{1}{1 - 3t} \right) \, dt = \int \frac{1}{t} \, dt + \int \frac{1}{1 - 3t} \, dt \)

The first integral is simply \( \ln |t| + C_1 \). For the second one, let \( u = 1 - 3t \), implying that \( du = -3 \, dt \) or \( dt = -\frac{1}{3} \, du \). Then

\[
\int \frac{1}{1 - 3t} \, dt = \int \frac{1}{u} \left( -\frac{1}{3} \right) \, du = -\frac{1}{3} \ln |u| + C_2 = -\frac{1}{3} \ln |1 - 3t| + C_2
\]

Therefore

\[
\int \left( t^{-1} + \frac{1}{1 - 3t} \right) \, dt = \ln |t| + C_1 - \frac{1}{3} \ln |1 - 3t| + C_2 = \ln \left| \frac{t}{(1 - 3t)^{1/3}} \right| + C
\]

(c)

\[
\int_{-\frac{2\pi}{3}}^{\frac{2}{3}} \frac{1}{4t^2 + 9} \, dt = \int_{-\frac{2\pi}{3}}^{\frac{2}{3}} \frac{1}{9 \left( \frac{4}{9} t^2 + 1 \right)} \, dt = \frac{1}{9} \int_{-\frac{2\pi}{3}}^{\frac{2}{3}} \frac{1}{\left( \frac{2}{3} \right)^2 + 1} \, dt
\]

Now let \( u = \frac{2t}{3} \) so that \( du = \frac{2}{3} \, dt \) \( \Rightarrow \) \( dt = \frac{3}{2} \, du \). Moreover, \( t = \frac{-3\sqrt{3}}{2} \) \( \Rightarrow \) \( u = -\sqrt{3} \); \( t = \frac{3}{2} \) \( \Rightarrow \) \( u = 1 \). Then

\[
\frac{1}{9} \int_{-3\sqrt{3}/2}^{3/2} \frac{1}{\left( \frac{2u}{3} \right)^2 + 1} \, du = \frac{1}{9} \int_{-\sqrt{3}}^{1} \frac{1}{u^2 + 1} \left( \frac{3}{2} \right) \, du = \frac{1}{6} \tan^{-1} u \bigg|_{-\sqrt{3}}^{1}
\]

\[
= \frac{1}{6} \left[ \tan^{-1} 1 - \tan^{-1} \left( -\sqrt{3} \right) \right] = \frac{1}{6} \left[ \frac{\pi}{4} - \left( -\frac{\pi}{3} \right) \right] = \frac{7\pi}{72}
\]

5. [10 pts] A culture initially contains 1000 bacteria and the number of bacteria increases at a rate proportional to the number present. After a third of an hour there are 3000 bacteria in the culture. When will the culture contain 81000 bacteria? Simplify your final answer so that it does not contain any logarithms.

**SOLUTION:**

Since the culture grows at a rate proportional to the number present, letting \( P(t) \) be the number of bacteria present at time \( t \) we have

\[
\frac{dP}{dt} = kP \quad \Rightarrow \quad P(t) = P(0)e^{kt}
\]

We are given that \( P(0) = 1000 \) and that \( P(1/3) = 3000 \) (if using \( t \) in minutes this is \( P(20) = 3000 \)). This allows us to find \( k \) as (left column for \( t \) in hours, right column for \( t \) in minutes)

Thus \( P(t) = 1000e^{kt} \) (or \( P(t) = 1000e^{(\ln 3^{1/20})t} \) if \( t \) is in minutes). We now need to find when the number of bacteria reaches 81000. This occurs when (left column for \( t \) in hours, right column for \( t \) in minutes)

\[
\begin{align*}
81000 &= 1000e^{(\ln 27)t} \\
81 &= e^{(\ln 27)t} \\
\ln 81 &= (\ln 27)t \\
\frac{\ln 81}{\ln 27} &= \frac{4\ln 3}{3} \\
t &= \frac{4}{3} \text{ hours}
\end{align*}
\]

\[
\begin{align*}
81000 &= 1000e^{(\ln 3^{1/20})t} \\
81 &= e^{(\ln 3^{1/20})t} \\
\ln 81 &= (\ln 3^{1/20})t \\
\frac{\ln 81}{\ln 3} &= \frac{20\ln 3}{\ln 3} = \frac{80\ln 3}{\ln 3} = 80 \text{ minutes}
\end{align*}
\]

6. [10 pts] In your bluebook, write \textbf{TRUE} if the statement is true and write \textbf{FALSE} if the statement is false. Do not abbreviate with T or F. No justification required and no partial credit given.

(a) \( \ln \sqrt{x+y} = \frac{1}{2} (\ln x + \ln y) \)

(b) \( \frac{d}{dx} \left( \frac{1}{2} \right)^{2x} = - \left( \frac{1}{2} \right)^{2x} \ln 4 \)

(c) \( \frac{d}{dx} \ln \ln x = 1 \)

(d) \( \int \pi^x \, dx = \pi^{x+1} + C \)

(e) \( \frac{d}{dx} \log x = \frac{1}{x \ln 10} \)

**SOLUTION:**

(a) \textbf{FALSE} \( \ln \sqrt{x+y} = \ln(x+y)^{1/2} = \frac{1}{2} \ln(x+y) \) but \( \ln(x+y) \neq \ln x + \ln y \)

(b) \textbf{TRUE} \( \frac{d}{dx} \left( \frac{1}{2} \right)^{2x} = \left( \frac{1}{2} \right)^{2x} \left( \ln 1 \right) \frac{d}{dx} (2x) = \left( \frac{1}{2} \right)^{2x} (2 \ln 2^{-1}) (2) = \left( \frac{1}{2} \right)^{2x} (-2 \ln 2) = - \left( \frac{1}{2} \right)^{2x} \ln 4 \)

(c) \textbf{TRUE} \( 3^{\ln x} = x \) and \( \frac{d}{dx} x = 1 \)

(d) \textbf{FALSE} \( \int \pi^x \, dx = \frac{\pi^x}{\ln \pi} + C \)

(e) \textbf{TRUE} \( \log x = \frac{\ln x}{\ln 10} \Rightarrow \frac{d}{dx} \log x = \frac{d}{dx} \left( \frac{\ln x}{\ln 10} \right) = \frac{1}{\ln 10} \frac{d}{dx} \ln x = \frac{1}{x \ln 10} \)