1. (15 points) Let \( f(x) = \frac{1}{x} \)

   (a) Estimate the area under the graph of \( f(x) \) from \( x = 1 \) to \( x = 4 \) with 6 rectangles and left endpoints. *You don’t have to simplify your answer.*

   (b) Sketch the curve and the approximating rectangles.

   (c) Is your estimate an underestimate or an overestimate? Why?

**SOLUTION**

(a) Here \( a = 1, b = 4 \) and \( n = 6 \), hence

\[
\Delta x = \frac{b - a}{n} = \frac{4 - 1}{6} = \frac{3}{6} = \frac{1}{2}.
\]

The relevant subintervals are 

\[
[1, 3/2] \quad [3/2, 2] \quad [2, 5/2] \quad [5/2, 3] \quad [3, 7/2] \quad [7/2, 4]
\]

and since we’re told to use lefthand endpoints, we choose

\[
x_1^* = 1 \quad x_2^* = 3/2 \quad x_3^* = 2 \quad x_4^* = 5/2 \quad x_5^* = 3 \quad x_6^* = 7/2
\]

so that the area under the curve is approximately

\[
L_6 = (f(1) + f(3/2) + f(2) + f(5/2) + f(3) + f(7/2)) \cdot \Delta x
\]

\[
= \left( 1 + \frac{2}{3} + \frac{1}{2} + \frac{2}{5} + \frac{1}{3} + \frac{1}{2} \right) \left( \frac{1}{2} \right)
\]

(b) The sketch looks something like this:

(c) This is an overestimate since the function is decreasing on the entire interval.
2. (15 points) The acceleration function (in m/s²) and the initial velocity are given for a particle moving along a line.

\[ a(t) = 2t - 1 \quad v(0) = -6 \]

(a) Find the velocity of the particle as a function of time.

(b) Find the distance traveled by the particle during the time interval \(1 \leq t \leq 4\). You don’t have to simplify your answer. Just get down to some number, even if it looks ugly!

**SOLUTION**

(a) To find velocity as a function of time, we start by finding the most general antiderivative of acceleration:

\[ v(t) = \int a(t) \, dt = \int (2t - 1) \, dt = t^2 - t + C. \]

To find the appropriate value of \(C\) we use the given data point that \(v(0) = -6\):

\[ v(0) = -6 \implies -6 = 0^2 - 0 + C \implies C = -6. \]

Hence \(v(t) = t^2 - t - 6\).

(b) To find the distance traveled during the time interval, we need to compute \(\int_1^4 |v(t)| \, dt\).

But to understand \(|v(t)|\) we need to analyze the sign of \(v(t)\). Note that \(v(t) = (t - 3)(t + 2)\), which is zero when \(t = -2\) and \(t = 3\), so we test accordingly:

<table>
<thead>
<tr>
<th>subinterval</th>
<th>sign of ((t - 3))</th>
<th>sign of ((t + 2))</th>
<th>sign of (v(t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-\infty, -2))</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>((-2, 3))</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>((3, \infty))</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
</tbody>
</table>

We only care about \(1 \leq t \leq 4\), but the table shows that \(v(t) \leq 0\) when \(1 \leq t \leq 3\) and \(v(t) \geq 0\) when \(3 \leq t \leq 4\).

Hence

\[
|v(t)| = \begin{cases} 
-v(t) & \text{if } 1 \leq t \leq 3 \\
 v(t) & \text{if } 3 \leq t \leq 4 
\end{cases}
\]
The total distance traveled by the particle is therefore

\[ \int_{1}^{4} |v(t)| \, dt = \int_{1}^{3} |v(t)| \, dt + \int_{3}^{4} |v(t)| \, dt \]

\[ = \int_{1}^{3} (-v(t)) \, dt + \int_{3}^{4} v(t) \, dt \]

\[ = \int_{1}^{3} -(t^2 - t - 6) \, dt + \int_{3}^{4} (t^2 - t - 6) \, dt \]

\[ = - \left[ \frac{1}{3}t^3 - \frac{1}{2}t^2 - 6t \right]_{1}^{3} + \left[ \frac{1}{3}t^3 - \frac{1}{2}t^2 - 6t \right]_{3}^{4} \]

\[ = - \left[ \left( \frac{1}{3} \cdot 3^3 - \frac{1}{2} \cdot 3^2 - 6 \cdot 3 \right) - \left( \frac{1}{3} \cdot 1^3 - \frac{1}{2} \cdot 1^2 - 6 \cdot 1 \right) \right] \]

\[ + \left[ \left( \frac{1}{3} \cdot 4^3 - \frac{1}{2} \cdot 4^2 - 6 \cdot 4 \right) - \left( \frac{1}{3} \cdot 3^3 - \frac{1}{2} \cdot 3^2 - 6 \cdot 3 \right) \right] \]

meters, and we can stop there.

3. (10 POINTS) A biologist investigating colony collapse disorder studied a population of honeybees for 6 months. She collected the following data about \( r(t) \), the rate of growth of the population (measured in bees per week), with time \( t \) measured in weeks:

<table>
<thead>
<tr>
<th>( t )</th>
<th>0</th>
<th>4</th>
<th>8</th>
<th>12</th>
<th>16</th>
<th>20</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r(t) )</td>
<td>100</td>
<td>2000</td>
<td>3000</td>
<td>11000</td>
<td>4000</td>
<td>1000</td>
<td>0</td>
</tr>
</tbody>
</table>

(a) What does \( \int_{0}^{24} r(t) \, dt \) represent? Answer with a sentence and include units.

(b) Using the given data, write down an overestimate and an underestimate to the quantity in part (a). *You don’t have to simplify your answer.*

**SOLUTION**

(a) The given integral represents the net change in the bee population during the 24 weeks in units of bees. We don’t really know that \( r(t) \) was always positive during those 24 weeks, so although it’s tempting, we’ll refrain from saying the increase in the bee population.

(b) To overestimate the given quantity, we pick the larger of the two data points on each subinterval. Hence

overestimate = \( (2000 + 3000 + 11000 + 11000 + 4000 + 1000)(4) \) bees

whereas to underestimate the given quantity, we pick the smaller of the two data points on each subinterval:

underestimate = \( (100 + 2000 + 3000 + 4000 + 1000 + 0)(4) \) bees
4. (15 POINTS) Please answer the following:

(a) Evaluate \( \int_{0}^{\pi/4} \frac{5 + \cos^{2}(\theta)}{\cos^{2}(\theta)} \, d\theta \)

(b) Find \( \frac{d}{dx} \left[ \int_{x^3}^{\pi/2} \sin \left( \frac{t}{2} \right) \cos \left( \frac{t}{3} \right) \, dt \right] \)

(c) Say \( g \) is a continuous function.

If \( \int_{0}^{5} g(x) \, dx = 10 \) and \( \int_{0}^{3} g(x) \, dx = 7 \) find \( \int_{3}^{5} 2g(x) \, dx \).

**SOLUTION**

(a) Rewrite the integrand:

\[
\int_{0}^{\pi/4} \frac{5 + \cos^{2}(\theta)}{\cos^{2}(\theta)} \, d\theta = \int_{0}^{\pi/4} \left( \frac{5}{\cos^{2}(\theta)} + \frac{\cos^{2}(\theta)}{\cos^{2}(\theta)} \right) \, d\theta \\
= \int_{0}^{\pi/4} (5 \sec^{2}(\theta) + 1) \, d\theta \\
= \left[ 5 \tan(\theta) + \theta \right]_{0}^{\pi/4} \\
= \left( 5 \tan \left( \frac{\pi}{4} \right) + \frac{\pi}{4} \right) - (\tan(0) + 0) \\
= 5 \cdot 1 + \frac{\pi}{4} - 0 \\
= 5 + \frac{\pi}{4}.
\]

(b)

\[
\frac{d}{dx} \left[ \int_{x^3}^{\pi/2} \sin \left( \frac{t}{2} \right) \cos \left( \frac{t}{3} \right) \, dt \right] = \frac{d}{dx} \left[ -\int_{\pi/2}^{x^3} \sin \left( \frac{t}{2} \right) \cos \left( \frac{t}{3} \right) \, dt \right] \quad \text{prop. of integrals}
\]

\[
= -\frac{d}{dx} \left[ \int_{\pi/2}^{x^3} \sin \left( \frac{t}{2} \right) \cos \left( \frac{t}{3} \right) \, dt \right] \quad \text{prop. of derivatives}
\]

\[
= -\sin \left( \frac{x^3}{2} \right) \cos \left( \frac{x^3}{3} \right) \cdot \frac{d}{dx} \left[ x^3 \right] \quad \text{FTC1 & chain rule}
\]

\[
= -\sin \left( \frac{x^3}{2} \right) \cos \left( \frac{x^3}{3} \right) (3x^2)
\]

(c) Since

\[
\int_{0}^{5} g(x) \, dx = \int_{0}^{3} g(x) \, dx + \int_{3}^{5} g(x) \, dx,
\]

filling in what we know gives

\[
10 = 7 + \int_{3}^{5} g(x) \, dx \quad \Rightarrow \quad \int_{3}^{5} g(x) \, dx = 10 - 7 = 3.
\]
Then

\[ \int_{3}^{5} 2g(x) \, dx = 2 \int_{3}^{5} g(x) \, dx = 2(3) = 6. \]

5. (23 POINTS)

(a) Determine a region whose area is equal to

\[ \lim_{n \to \infty} \frac{3}{n} \sum_{k=1}^{n} \sqrt{1 + \frac{3k}{n}} \]

Your answer should include the relevant function, an interval and a labeled sketch of the region. You don’t have to evaluate the limit.

(b) Write down a definite integral that equals the given limit from part (a). You don’t have to evaluate the integral.

(c) What is the average value of the function you found in part (a) over the interval you found in part (a)? Simplify your answer.

(d) Apply the Mean Value Theorem for Integrals to your function over your interval, finding the \( x \)-value(s) that your book calls \( c \).

(e) On top of your graph from part (a), sketch a rectangle whose area is the same as the area of the region from part (a).

**SOLUTION**

(a) Think

\[ A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*) \Delta x = \lim_{n \to \infty} \sum_{k=1}^{n} \sqrt{1 + \frac{3k}{n}} \cdot \frac{3}{n}, \]

where we pulled the \( \frac{3}{n} \) into the sum.

The key is that the \( 1 + \frac{3k}{n} \) should remind you of right-hand endpoints, \( x_k^* = a + k \Delta x \).

In that case, \( a = 1 \) and \( \Delta x = \frac{b-a}{n} = \frac{3}{n} \) which forces \( b = 4 \).

The relevant function is then \( f(x) = \sqrt{x} \) and the interval is \([1, 4]\).

The first sketch should look like this:
(b) $\int_{1}^{4} \sqrt{x} \, dx$

(c) Rewriting $\sqrt{x} = x^{1/2}$,
\[
f_{ave} = \frac{1}{b-a} \int_{a}^{b} f(x) \, dx
= \frac{1}{4-1} \int_{1}^{4} x^{1/2} \, dx
= \frac{1}{3} \left[ \frac{2}{3} x^{3/2} \right]_{1}^{4}
= \frac{1}{3} \cdot \frac{2}{3} \left[ 4^{3/2} - 1^{3/2} \right]
= \frac{2}{9} [8 - 1]
= \frac{2}{9} \cdot 7
= \frac{14}{9}.
\]

(d) Since $f(x) = \sqrt{x}$ is continuous on the interval [1, 4],
the Mean Value Theorem for Integrals guarantees an $x$-value, call it $c$, for which $f_{ave} = f(c)$.
So we solve
\[
\frac{14}{9} = f(c) \implies \frac{14}{9} = \sqrt{c} \implies c = \left( \frac{14}{9} \right)^2.
\]
Note that
\[
1 = \frac{9}{9} < \frac{14}{9} < \frac{18}{9} = 2 \implies 1 = 1^2 < \left( \frac{14}{9} \right)^2 < 2^2 = 4,
\]
which is to say that our $c$ is actually in the interval [1, 4].

(e) The new graph should look like this:
6. (22 points) Let \( G(x) = \int_{-3}^{x} f(t) \, dt \) for \(-3 \leq x \leq 3\), where the graph of \( y = f(t) \) is shown below.

Please answer the following questions, briefly justifying your answers:

(a) What is \( G(-3) \)?
(b) What is \( G(-1) \)?
(c) What is \( G(0) \)?
(d) What is \( G(2) \)?
(e) What is \( G'(-1) \)?
(f) What is \( G'(2) \)?
(g) What is \( G''(1) \)?
(h) Where does \( G \) have a relative minimum? Why?
(i) Does \( G \) have any inflection points? Where?
(a) \( G(-3) = \int_{-3}^{-3} f(t) \, dt = 0 \)

(b) \( G(-1) = \int_{-3}^{-1} f(t) \, dt = -2(2) = -4 \) (it’s a 2 \times 2 rectangle underneath the axis)

(c) \( G(0) = \int_{-3}^{-1} f(t) \, dt + \int_{-1}^{0} f(t) \, dt = -4 - \frac{1}{2}(1)(2) = -4 - 1 = -5 \) (the rectangle from part (b) with an additional triangle under the axis)

(d) \( G(2) = \int_{-3}^{0} f(t) \, dt + \int_{0}^{2} f(t) \, dt = -5 + \frac{1}{2}(2)(4) = -5 + 4 = -1 \), using the triangle above the axis with base width 2 and height 4 in between \( t = 0 \) and \( t = 2 \) and the answer from part (c)

By FTC1, note that \( G'(x) = \frac{d}{dx} \left[ \int_{-3}^{x} f(t) \, dt \right] = f(x) \), valid for \(-3 < x < 3\). Hence

(e) \( G'(-1) = f(-1) = -2 \) (just look at the graph of \( f \))

(f) \( G'(2) = f(2) = 4 \)

Since \( G'(x) = f(x) \), \( G''(x) = f'(x) \)

(g) So \( G''(1) = f'(1) = 2 \), which is the slope of the linear piece of \( f \) between \( t = -1 \) and \( t = 2 \).

(h) \( G \) has a relative minimum at zero by the first derivative test, since \( G' = f \) switches sign from negative to positive there.

(i) Since \( G'' = f' > 0 \) (i.e. \( f \) is increasing) to the left of 2 and \( G'' = f' < 0 \) (i.e. \( f \) is decreasing) to the right of 2, \( G \) switches from concave up to concave down at 2, hence there is an inflection point at \( x = 2 \).