On the front of your bluebook, please write: a grading key, your name, student ID, your lecture number and instructor. This exam is worth 150 points and has 7 questions.

- Submit this exam sheet with your bluebook. However, nothing on this exam sheet will be graded. Make sure all of your work is in your bluebook.
- **Show all work and simplify your answers!** Answers with no justification will receive no points unless otherwise noted. **Please begin each problem on a new page.**
- You will be taking this exam in a proctored and honor code enforced environment. This means: no notes or papers, calculators, cell phones, or other electronic devices are permitted.

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1. [30 pts] Evaluate the following limits. Be sure to show all your work.

   (a) $\lim_{x\to\pi/8} \sin 2x \cos 2x \tan 2x$

   (b) $\lim_{t\to-2^+} \dfrac{t^2-3t-10}{t^3+5t^2+6t}$

   (c) $\lim_{x\to\infty} \dfrac{1-x^3}{3x+2}$

   (d) $\lim_{\theta\to 0} \dfrac{\tan\theta}{\theta}$

   (e) $\lim_{h\to 0} \dfrac{\sqrt{9+h}-3}{h}$

   <u>**SOLUTION:**</u>

   (a) Direct substitution yields

$$\lim_{x\to\pi/8} \sin 2x\cos 2x\tan 2x = \sin\left(\frac{2\pi}{8}\right)\cos\left(\frac{2\pi}{8}\right)\tan\left(\frac{2\pi}{8}\right) = \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right)$$
$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)(1) = \frac{1}{2}$$

   (b) Direct substitution yields the indeterminate form $0/0$ so we need to do some further analysis.

$$\lim_{t\to-2^+}\frac{t^2-3t-10}{t^3+5t^2+6t} = \lim_{t\to-2^+}\frac{(t-5)\cancel{(t+2)}}{t(t+3)\cancel{(t+2)}} = \lim_{t\to-2^+}\frac{t-5}{t(t+3)} = \frac{-2-5}{(-2)(-2+3)} = \frac{-7}{-2} = \frac{7}{2}$$

   (c) As x becomes large, the numerator becomes arbitrarily small and the denominator becomes arbitrarily large. So multiply by a convenient form of 1.

$$\lim_{x\to\infty}\frac{1-x^3}{3x+2}\left(\frac{1/x}{1/x}\right) = \lim_{x\to\infty}\frac{\frac{1}{x}-x^2}{3+\frac{2}{x}} = -\infty \qquad \text{Note: } \frac{1}{x}\to 0, \frac{2}{x}\to 0 \text{ as } x\to\infty$$

(d) Use the fact that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

$$\lim_{\theta\to 0}\frac{\tan\theta}{\theta} = \lim_{\theta\to 0}\frac{\frac{\sin\theta}{\cos\theta}}{\theta} = \lim_{\theta\to 0}\frac{\sin\theta}{\theta\cos\theta} = \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)\left(\frac{1}{\cos\theta}\right)$$

$$= \left(\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\right)\left(\lim_{\theta\to 0}\frac{1}{\cos\theta}\right) = 1\left(\frac{1}{\lim\limits_{\theta\to 0}\cos\theta}\right) = \frac{1}{\cos 0} = 1$$

(e) Direct substitution yields the indeterminate form $0/0$. Multiply by a convenient form of 1 using the conjugate of the numerator.

$$\lim_{h\to 0}\frac{\sqrt{9+h}-3}{h}\left(\frac{\sqrt{9+h}+3}{\sqrt{9+h}+3}\right) = \lim_{h\to 0}\frac{\cancel{9}+\cancel{h}-\cancel{9}}{\cancel{h}\left(\sqrt{9+h}+3\right)} = \lim_{h\to 0}\frac{1}{\sqrt{9+h}+3} = \frac{1}{6}$$

Note that this is just the definition of $f'(9)$ where $f(x) = \sqrt{x}$.

$\blacksquare$

2. [40 pts] The following problems are unrelated.

(a) Find the linearization of $y = \left(\dfrac{x+5}{x^2+2}\right)^2$ at $a = 1$.

(b) Find $f''(\pi)$ if $f(x) = x\tan x$.

(c) Let $y = 1 - |x-2|$.

i. Write y as a piecewise function.

ii. Graph y, labeling all intercepts.

iii. Using your graph, find all x values, if any, where y is not differentiable.

(d) Find the absolute extrema, if any exist, of the function $g(x) = \sqrt[3]{x^2}\,(2\sqrt[3]{x}-3)$ on $[-1,1]$.

<u>SOLUTION:</u>

(a)

$$y' = 2\left(\frac{x+5}{x^2+2}\right)\left[\frac{(x^2+2)(1)-(x+5)(2x)}{(x^2+2)^2}\right] = 2\left(\frac{x+5}{x^2+2}\right)\left[\frac{2-10x-x^2}{(x^2+2)^2}\right] = \frac{2(x+5)(2-10x-x^2)}{(x^2+2)^3}$$

Thus $y'(1) = \dfrac{2(1+5)\left[2-10(1)-1^2\right]}{(1^2+2)^3} = \dfrac{2(6)(-9)}{3^3} = \dfrac{2(2)(3)(-3^2)}{3^3} = -4$. Evaluating the function at $x = 1$ yields $\left(\dfrac{1+5}{1^2+2}\right)^2 = 4$, giving the linearization $L(x) = 4 - 4(x-1) = -4x + 8$.

(b)

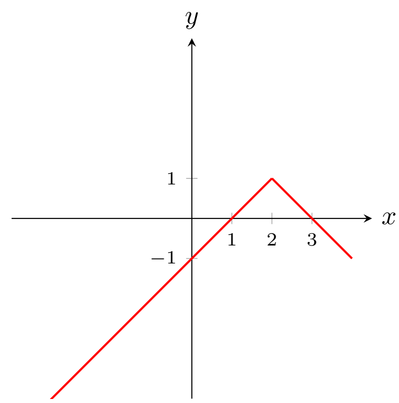
$$\begin{aligned} f'(x) &= x(\tan x)' + \tan x(x)' = x\sec^2 x + \tan x(1) = x\sec^2 x + \tan x \\ f''(x) &= x(\sec^2 x)' + \sec^2 x(x)' + (\tan x)' = x(2\sec x)(\sec x)' + \sec^2 x(1) + \sec^2 x \\ &= 2x\sec x(\sec x\tan x) + \sec^2 x + \sec^2 x = 2x\sec^2 x\tan x + 2\sec^2 x = 2\sec^2 x(x\tan x + 1) \end{aligned}$$

Thus $f''(\pi) = 2\sec^2\pi(\pi\tan\pi + 1) = 2(-1)^2[\pi(0)+1] = 2$

(c) i. If $x-2\geq 0$ (equivalently $x\geq 2$), then $|x-2|=x-2$ and $1-|x-2|=1-(x-2)=3-x$. If $x-2<0$ (equivalently $x<2$) then $|x-2|=-(x-2)=2-x$ and $1-|x-2|=1-(2-x)=x-1$. Thus written as a piecewise function

$$y=1-|x-2|=\begin{cases}3-x & x\geq 2\\ x-1 & x<2\end{cases}$$

ii. Sketch.



iii. y is not differentiable at $x=2$ where a cusp exists (slope to the left of $x=2$ is 1 whereas to the right of $x=2$ the slope is -1).

(d) Begin by rewriting and simplifying g as

$$g(x)=\sqrt[3]{x^2}\left(2\sqrt[3]{x}-3\right)=x^{2/3}\left(2x^{1/3}-3\right)=2x-3x^{2/3}$$

$g(x)$ is continuous for all real numbers since it is the sum of two functions that are continuous for all real numbers. Thus, $g(x)$ is continuous on $[-1,1]$ which is a closed and bounded interval. Consequently, the Extreme Value Theorem states that g will attain a maximum and a minimum somewhere on $[-1,1]$. Checking the endpoints of the interval, $g(-1)=2(-1)-3(-1)^{2/3}=-5$ and $g(1)=2(1)-3(1)^{2/3}=-1$. Now

$$\frac{\mathrm{d}g}{\mathrm{d}x}=2-2x^{-1/3}=2(1-x^{-1/3})=2\left(1-\frac{1}{x^{1/3}}\right)=2\left(\frac{x^{1/3}-1}{x^{1/3}}\right)$$

From this we see that the critical points of g are $x=0$, where the derivative fails to exist (but the function is defined), and $x=1$ where the derivative equals 0. We have already evaluated $g(1)$, and $g(0)=0$. Thus the absolute maximum of g is $g(0)=0$ and the absolute minimum is $g(-1)=-5$.

Note: Had we not simplified g prior to differentiation, we have, using the product rule,

$$\begin{aligned}\frac{\mathrm{d}g}{\mathrm{d}x}&=x^{2/3}\frac{\mathrm{d}}{\mathrm{d}x}\left(2x^{1/3}-3\right)+\left(2x^{1/3}-3\right)\frac{\mathrm{d}}{\mathrm{d}x}x^{2/3}=x^{2/3}\left(\frac{2}{3}x^{-2/3}\right)+\left(2x^{1/3}-3\right)\left(\frac{2}{3}x^{-1/3}\right)\\&=\frac{2}{3}+\frac{4}{3}-2x^{-1/3}=2-2x^{-1/3}\end{aligned}$$

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3. [10 pts] The height of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of 2 cm^2/min. At what rate is the base of the triangle changing when the height is 10 cm and the area is 100 cm^2?

SOLUTION:

Let A be the area of the triangle, b its base and h its height. We are given $\frac{\mathrm{d}A}{\mathrm{d}t} = 2$, $\frac{\mathrm{d}h}{\mathrm{d}t} = 1$ and we are asked to find $\frac{\mathrm{d}b}{\mathrm{d}t}$ when $h = 10$ and $A = 100$. The area of a triangle is $A = \frac{1}{2}bh$. Differentiating this equation yields

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}\left(b\frac{\mathrm{d}h}{\mathrm{d}t} + h\frac{\mathrm{d}b}{\mathrm{d}t}\right) \implies \frac{\mathrm{d}b}{\mathrm{d}t} = \frac{1}{h}\left(2\frac{\mathrm{d}A}{\mathrm{d}t} - b\frac{\mathrm{d}h}{\mathrm{d}t}\right)$$

Everything on the right hand side of the last equation is known except for b. However b can be determined from the area formula as $b = \frac{2A}{h}$. Consequently

$$\frac{\mathrm{d}b}{\mathrm{d}t} = \frac{1}{h}\left(2\frac{\mathrm{d}A}{\mathrm{d}t} - \frac{2A}{h}\frac{\mathrm{d}h}{\mathrm{d}t}\right) = \frac{1}{10}\left[2(2) - \frac{(2)(100)}{10}(1)\right] = \frac{1}{10}(4-20) = -\frac{8}{5}\text{ cm/min} = -1.6\text{ cm/min}$$

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4. [15 pts] Consider the function $f(x) = x\sqrt{5-x}$.

   (a) Find the domain of $f(x)$, writing your answer using interval notation.

   (b) Find the average rate of change of $f(x)$ on the interval $[1, 5]$. What geometric property of the graph of f does the average rate of change represent?

   (c) Is there is a point in the interval $(1, 5)$ where the instantaneous rate of change of f equals its average rate of change over that interval? Justify your answer.

SOLUTION:

(a) For the square root to be defined, we need $5 - x \geq 0 \implies 5 \geq x$. Thus the domain is $(-\infty, 5]$.

(b) The average rate of change of $f(x)$ is given by

$$\frac{f(5) - f(1)}{5-1} = \frac{5\sqrt{5-5} - 1\sqrt{5-1}}{4} = -\frac{1}{2}$$

and geometrically represents the slope of the secant line between the points $(1, f(1))$ and $(5, f(5))$.

(c) The function x is continuous for all real numbers and $\sqrt{5-x}$ is continuous on $(-\infty, 5]$. Thus $f(x)$ is continuous on $(-\infty, 5]$ since it is the product of two continuous functions and hence is continuous on $[1, 5]$. Furthermore,

$$f'(x) = \frac{-x}{2\sqrt{5-x}} + \sqrt{5-x}$$

which exists on $(-\infty, 5)$ and thus on the interval $(1, 5)$. Both hypotheses of the Mean Value Theorem are satisfied, guaranteeing that a c exists in $(1, 5)$ such that

$$f'(c) = \frac{f(5) - f(1)}{5-1},$$

that is, there is a point in $(1, 5)$ where the instantaneous rate of change of f equals its average rate of change over the interval.

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5. [25 pts] Suppose the position of a particle moving horizontally after t hours is given by $s(t) = \cos^2 t$ for $0 \leq t \leq 2\pi$. Let s be measured in miles with $s > 0$ corresponding to positions to the right of the origin.

(a) Will the object ever be to the left of the origin? Briefly explain.

(b) Find the velocity of the object at time t.

(c) What time(s), if any, is the object at rest?

(d) Find the total distance traveled by the object.

SOLUTION:

(a) No. The position function is never negative ($\cos^2 t \geq 0$).

(b) $v(t) = s'(t) = 2\cos t(-\sin t) = -2\sin t \cos t$

(c) The object is at rest when its velocity vanishes. This occurs if $-2\sin t \cos t = 0$ or when $\sin t = 0 \implies t = 0, \pi, 2\pi$ or when $\cos t = 0 \implies t = \pi/2, 3\pi/2$. The object is at rest when $t = 0, \pi/2, \pi, 3\pi/2$, and 2π.

(d) Since the object is moving both to the left and to the right during the given interval, this will need to be considered when determining the total distance traveled.

$$\begin{aligned}\text{Distance} &= |s(\pi/2) - s(0)| + |s(\pi) - s(\pi/2)| + |s(3\pi/2) - s(\pi)| + |s(2\pi) - s(3\pi/2)| \\ &= |\cos^2(\pi/2) - \cos^2(0)| + |\cos^2(\pi) - \cos^2(\pi/2)| \\ &\qquad + |\cos^2(3\pi/2) - \cos^2(\pi)| + |\cos^2(2\pi) - \cos^2(3\pi/2)| \\ &= |0-1| + |1-0| + |0-1| + |1-0| = 4 \text{ miles}\end{aligned}$$

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6. [15 pts] Consider the relation $x^2 + y^2 + 6x - 4y = -12$.

(a) Draw the graph of this relation, labeling important points. (Hint: It is a circle)

(b) Find $\mathrm{d}y/\mathrm{d}x$.

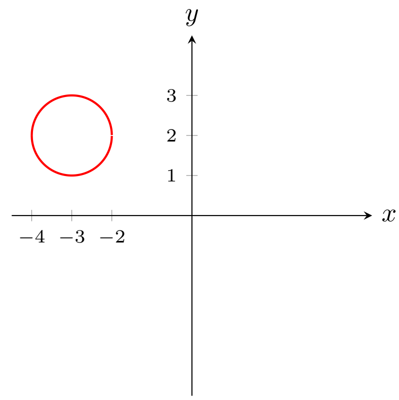
(c) Find the point(s) on the graph where the slope of the tangent line is -1.

SOLUTION:

(a) Complete the square in x and y to give

$$x^2 + 6x + 9 - 9 + y^2 - 4y + 4 - 4 = -12 \implies (x+3)^2 + (y-2)^2 - 13 = -12 \implies (x+3)^2 + (y-2)^2 = 1$$

which is a circle of radius 1 centered at $(-3, 2)$.



(b) Beginning with the original relation we have, using implicit differentiation,

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} + 6 - 4\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \implies x + 3 + (y-2)\frac{\mathrm{d}y}{\mathrm{d}x} \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-(x+3)}{y-2} = \frac{x+3}{2-y}$$

(c) We want the slope of the tangent line to be -1. This requires

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+3}{2-y} = -1 \implies x + 3 = -1(2-y) \implies y = x + 5$$

To find the points on the graph where this occurs, substitute this value for y into the equation obtained after completing the square to make the algebra a bit easier.

$$(x+3)^2 + (x+5-2)^2 = 1 \implies (x+3)^2 + (x+3)^2 = 1 \implies 2(x+3)^2 = 1 \implies x + 3 = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$$

Thus

$$x = -3 \pm \frac{\sqrt{2}}{2} \quad \text{and} \quad y = \left(-3 \pm \frac{\sqrt{2}}{2}\right) + 5 = 2 \pm \frac{\sqrt{2}}{2}$$

and the points on the graph where the slope of the tangent line is -1 are

$$\left(-3 + \frac{\sqrt{2}}{2}, 2 + \frac{\sqrt{2}}{2}\right) \quad \text{and} \quad \left(-3 - \frac{\sqrt{2}}{2}, 2 - \frac{\sqrt{2}}{2}\right)$$

Note: Had $y = x + 5$ been used in the original relation the equation $2x^2 + 12x + 17 = 0$ would have resulted, which can be solved using the quadratic formula.

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7. [15 pts] In your bluebook, write **T** if the statement is true and write **F** is the statement is false. No justification required and no partial credit given.

(a) $\cos^2\sqrt{x} + \sin^2\sqrt{x} = 1$

(b) Local extrema of a function occur at all of the function's critical points.

(c) $q(a) = 0$ guarantees that the line $x = a$ is a vertical asymptote of the function $r(x) = \dfrac{p(x)}{q(x)}$.

(d) If $\lim\limits_{x\to a} f(x) \neq f(a)$, then $\lim\limits_{x\to a} \dfrac{f(x) - f(a)}{x - a}$ does not exist.

(e) If a function $f(x)$ is continuous on the interval $[a, b]$ and $f(a) = f(b)$, then f must possess a horizontal tangent at some point c in (a, b).

SOLUTION:

(a) **TRUE** A fundamental trigonometric identity

(b) **FALSE** Consider $f(x) = x^3$. Then $x = 0$ is a critical point of f $[f'(0) = 0]$ but $f(0)$ is not a local extremum.

(c) **FALSE** If $p(a) = 0$ as well, then $r(x)$ possesses a removable discontinuity at $x = a$, not a vertical asymptote.

(d) **TRUE** If f is not continuous at a then it is not differentiable there.

(e) **FALSE** Consider $f(x) = |x|$ on $[-1, 1]$. If f is also differentiable on (a, b), then Rolle's Theorem applies and f will indeed have a horizontal tangent somewhere in (a, b).

■