APPM 1340 **Exam 3** **Fall 2017**

1. [24 pts] For the given function, find the indicated derivative. Simplify your final answers, writing them without fractional and/or negative exponents.

a. $p(t) = \dfrac{1}{\sqrt{3-t}} + 2\pi^7,\ \ p'(t)$ b. $y(\theta) = \sin(\cos 6\theta),\ \ y'\left(\dfrac{\pi}{4}\right)$ c. $f(x) = x\sqrt{x^2+2},\ \ f'(x)$

SOLUTION:

(a) $p(t) = (3-t)^{-1/2} + 2\pi^7 \implies p'(t) = -\dfrac{1}{2}(3-t)^{-3/2}(-1) + 0 = \dfrac{1}{2(3-t)^{3/2}} = \dfrac{1}{2\sqrt{(3-t)^3}}$

(b) $y'(\theta) = \cos(\cos 6\theta)(\cos 6\theta)' = \cos(\cos 6\theta)(-\sin 6\theta)(6\theta)' = -6(\sin 6\theta)\cos(\cos 6\theta))$ implying that

$$y'\left(\frac{\pi}{4}\right) = -6\sin\left(\frac{6\pi}{4}\right)\cos\left[\cos\left(\frac{6\pi}{4}\right)\right] = -6\sin\left(\frac{3\pi}{2}\right)\cos\left[\cos\left(\frac{3\pi}{2}\right)\right] = -6(-1)\cos 0 = 6$$

(c) We can rewrite the original function as $f(x) = x(x^2+2)^{1/2}$ so that

$$f'(x) = x\left[\frac{1}{2}(x^2+2)^{-1/2}(2x)\right] + (x^2+2)^{1/2}(1) = x^2(x^2+2)^{-1/2} + (x^2+2)^{1/2}$$
$$= (x^2+2)^{-1/2}\left[x^2 + (x^2+2)\right] = (x^2+2)^{-1/2}(2x^2+2) = \frac{2(x^2+1)}{\sqrt{x^2+2}}$$

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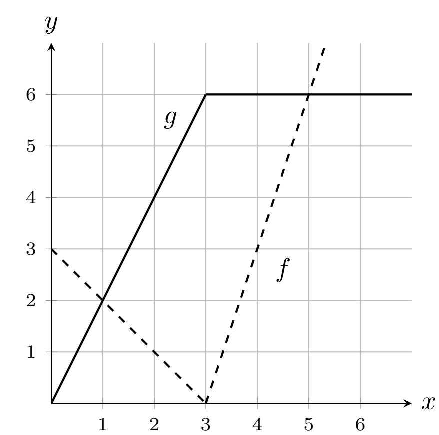
2. [12 pts] f and g are functions whose graphs are shown in the figure below. Use these to find the following.

a. $f'(2)$ b. $g'(2)$ c. $P'(2)$, where $P(x) = \dfrac{f(x)}{g(x)}$



SOLUTION:

(a) $f'(2)$ is the slope of the graph of f at $x = 2$, which is $\dfrac{\Delta y}{\Delta x} = \dfrac{0-3}{3-0} = -1$

(b) $g'(2)$ is the slope of the graph of g at $x = 2$, which is $\dfrac{\Delta y}{\Delta x} = \dfrac{6-0}{3-0} = 2$

(c) Using the quotient rule,

$$P'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \implies P'(2) = \frac{g(2)f'(2) - f(2)g'(2)}{[g(2)]^2} = \frac{(4)(-1) - (1)(2)}{4^2} = -\frac{6}{16} = -\frac{3}{8}$$

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3. [12 pts] Consider the relation $x^2 - xy + y^2 = 3$.

(a) Find $\mathrm{d}y/\mathrm{d}x$.

(b) Find all points on the graph where the tangent line is horizontal.

(c) Find all points on the graph where the tangent line is vertical.

SOLUTION:

(a) $2x - xy' - y + 2yy' = 0 \implies (2y - x)y' = y - 2x \implies y' = \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y - 2x}{2y - x}$

(b) Horizontal tangents occur where $y' = 0$, in this case, when $y - 2x = 0$ or $y = 2x$. Substituting this into the original relation yields $x^2 - x(2x) + (2x)^2 = 3 \implies 3x^2 = 3 \implies x = \pm 1$. Since $y = 2x$, horizontal tangents occur at $(1, 2)$ and $(-1, -2)$. (Note that the denominator of y' does not equal zero at either of these points.)

(c) Vertical tangents occur where y' does not exist. In this case, when $2y - x = 0$ or $2y = x$. Substituting this into the original relation yields $(2y)^2 - 2y(y) + y^2 = 3 \implies 3y^2 = 3 \implies y = \pm 1$. Since $x = 2y$, vertical tangents occur at $(2, 1)$ and $(-2, -1)$. (Note that the numerator of y' does not equal zero at either of these points.)

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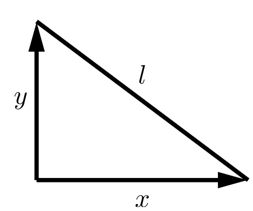
4. [12 pts] Two hikers start moving from the same point. One travels north at 1.5 mph and the other travels east at 2 mph. At what rate is the distance between the hikers increasing two hours later?

SOLUTION:



Using the notation in the figure, we are given $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 2$ and $\dfrac{\mathrm{d}y}{\mathrm{d}t} = 1.5$ and we are asked to find $\dfrac{\mathrm{d}l}{\mathrm{d}t}$. From the Pythagorean Theorem $x^2 + y^2 = l^2$ which, upon differentiation, yields

$$2x\frac{\mathrm{d}x}{\mathrm{d}t} + 2y\frac{\mathrm{d}y}{\mathrm{d}t} = 2l\frac{\mathrm{d}l}{\mathrm{d}t} \implies \frac{\mathrm{d}l}{\mathrm{d}t} = \frac{1}{l}\left(x\frac{\mathrm{d}x}{\mathrm{d}t} + y\frac{\mathrm{d}y}{\mathrm{d}t}\right)$$

After 2 hours, the eastbound hiker has gone $(2 \text{ mph})(2 \text{ hours}) = 4$ miles $= x$ while the northbound hiker has gone $(1.5 \text{ mph})(2 \text{ hours}) = 3$ miles $= y$. Consequently, at this time $l = \sqrt{3^2 + 4^2} = 5$ and

$$\frac{\mathrm{d}l}{\mathrm{d}t} = \frac{1}{5}\left[(4)(2) + 3\left(\frac{3}{2}\right)\right] = \frac{5}{2} \text{ mph}$$

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5. [12 pts] Find an equation of the normal line to the parabola $y = x^2 - 9x + 8$ that is parallel to the line $x - 7y - 3 = 0$. Put your answer in slope/intercept form.

SOLUTION:

$y' = 2x - 9$ gives the slope of the line tangent to the parabola at the point x. The slope of the normal line at this point is the negative reciprocal of this, namely $\frac{1}{9-2x}$. We want this normal line to be parallel to the given line, thus its slope must be the same as that of the given line. Now $x - 7y - 3 = 0 \implies 7y = x - 3 \implies y = \frac{1}{7}(x-3)$ so the given line's slope is $\frac{1}{7}$. Therefore we have

$$\frac{1}{9-2x} = \frac{1}{7} \implies 9 - 2x = 7 \implies x = 1$$

The y-value corresponding to $x = 1$ is $y = 1^2 - 9(1) + 8 = 0$. So the normal line to the parabola at the point $(1, 0)$ will be parallel to the given line. The normal line's equation is thus $y - 0 = \frac{1}{7}(x-1) \implies y = \frac{1}{7}x - \frac{1}{7}$.

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6. [28 pts] Suppose the position of an object moving horizontally after t seconds is given by $s(t) = -t^3 + 6t^2 - 9t$ for $0 \le t \le 4$. s is measured in feet with $s > 0$ corresponding to positions to the right of the origin.

(a) Find the velocity at time t.

(b) Find the acceleration at time t.

(c) When is the object not moving?

(d) When is the object moving to the left?

(e) When is the object moving to the right?

(f) Where is the object at the end of the time interval?

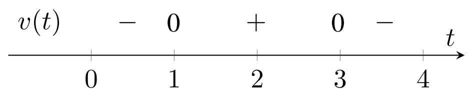
(g) What is the total distance that the object has traveled?

SOLUTION:

(a) $v(t) = s'(t) = -3t^2 + 12t - 9 = -3(t^2 - 4t + 3) = -3(t-1)(t-3)$

(b) $a(t) = v'(t) = s''(t) = -6t + 12 = -6(t-2)$

(c) The object is not moving when its velocity is zero. This occurs when $v(t) = -3(t-1)(t-3) = 0$, that is, when $t = 1$ or 3 seconds.



(d) The object is moving to the left when its velocity is negative. From the above figure, this occurs for $0 \leq t < 1$ or $3 < t \leq 4$, which can be written as $t \in [0, 1) \cup (3, 4]$.

(e) The object is moving to the right when its velocity is positive. From the above figure, this occurs for $1 < t < 3$, which can can be written as $t \in (1, 3)$.

(f) To simplify the position calculations, rewrite the position function in factored form as $s(t) = -t(t-3)^2$. At the end of the time interval, $t = 4$, the object is at $s(4) = -4(4-3)^2 = -4$, 4 feet to the left of the origin.

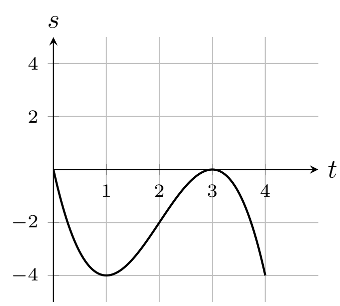
(g) To obtain the total distance traveled, we need to consider the direction of movement.

On the interval $0 \leq t \leq 1$, distance traveled to the left is $|s(1) - s(0)| = |-4 - 0| = 4$

On the interval $1 \leq t \leq 3$, distance traveled to the right is $|s(3) - s(1)| = |0 - (-4)| = 4$

On the interval $3 \leq t \leq 4$, distance traveled to the left is $|s(4) - s(3)| = |-4 - 0| = 4$

Thus, the total distance traveled is 12 feet. A graph of the position function follows.



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