1. [12 pts] In your bluebook, write **T** if the statement is true and write **F** if the statement is false. No justification required and no partial credit given.

   (a) If x is any real number, then $\sqrt{x^2} = x$.

   (b) If $r(x) = \dfrac{p(x)}{q(x)}$ is a rational function and $q(b) \neq 0$, then $\lim\limits_{x \to b} r(x) = r(b)$.

   (c) All continuous functions are differentiable for all x values in their domain.

   (d) Let $f(x)$ be a function such that $f(a) < 0 < f(b)$ with $a < b$. Then the graph of f must cross the x-axis somewhere between a and b.

   SOLUTION:

   (a) **FALSE** $\sqrt{x^2} = |x|$

   (b) **TRUE** Rational functions are continuous on their domains.

   (c) **FALSE** Consider $f(x) = |x|$. This function is continuous on $\mathbb{R}$ but is not differentiable at $x = 0$.

   (d) **FALSE** $f(x)$ must be continuous for there to be an x-intercept guaranteed by the Intermediate Value Theorem.

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2. [15 pts] Let $g(x) = \dfrac{x^3 + 5x^2 + 6x}{2x^3 - 2x^2 - 4x}$.

   (a) Write the domain of $g(x)$ in interval notation.

   (b) Find all horizontal asymptotes of $g(x)$. Justify your answer using limits.

   (c) Find all vertical asymptotes of $g(x)$. Justify your answer using limits.

   SOLUTION:
   For the various parts of this problem, it is useful to consider other forms of the function: factored, and rewritten (assuming $x \neq 0$) after multiplication by $(1/x^3)/(1/x^3)$. Thus

$$g(x) = \frac{x^3 + 5x^2 + 6x}{2x^3 - 2x^2 - 4x} = \frac{x(x+2)(x+3)}{2x(x+1)(x-2)} = \frac{1 + \frac{5}{x} + \frac{6}{x^2}}{2 - \frac{2}{x} - \frac{4}{x^2}}$$

   (a) From the factored form, the zeros of the denominator are $-1, 0, 2$. Therefore the domain of $g(x)$ is

$$(-\infty, -1) \cup (-1, 0) \cup (0, 2) \cup (2, \infty)$$

   (b) To check for horizontal asymptotes, compute limits at both positive and negative infinity.

$$\lim_{x \to \infty} \frac{x^3 + 5x^2 + 6x}{2x^3 - 2x^2 - 4x} = \lim_{x \to \infty} \frac{1 + \frac{5}{x} + \frac{6}{x^2}}{2 - \frac{2}{x} - \frac{4}{x^2}} = \frac{1}{2} \quad \text{and} \quad \lim_{x \to -\infty} \frac{x^3 + 5x^2 + 6x}{2x^3 - 2x^2 - 4x} = \lim_{x \to -\infty} \frac{1 + \frac{5}{x} + \frac{6}{x^2}}{2 - \frac{2}{x} - \frac{4}{x^2}} = \frac{1}{2}$$

   This implies that $y = \frac{1}{2}$ is a (the only) horizontal asymptote.

   (c) Possible candidates for vertical asymptotes are $x = -1, 0, 2$. To prove which are and are not vertical asympstotes, it is sufficient to compute one-sided limits. Those limits that are infinite prove that the candidate is indeed a vertical asymptote.

$$\lim_{x \to 0^-} \frac{x^3 + 5x^2 + 6x}{2x^3 - 2x^2 - 4x} = \lim_{x \to 0^-} \frac{(x+2)(x+3)}{2(x+1)(x-2)} = \frac{(2)(3)}{2(1)(-2)} = -\frac{3}{2} \tag{1}$$

$$\lim_{x \to 0^+} \frac{x^3 + 5x^2 + 6x}{2x^3 - 2x^2 - 4x} = \lim_{x \to 0^+} \frac{(x+2)(x+3)}{2(x+1)(x-2)} = \frac{(2)(3)}{2(1)(-2)} = -\frac{3}{2} \tag{2}$$

$$\lim_{x \to -1^+} \frac{x^3 + 5x^2 + 6x}{2x^3 - 2x^2 - 4x} = \lim_{x \to -1^+} \frac{(x+2)(x+3)}{2(x+1)(x-2)} = -\infty \quad \left(\frac{(1)(2)}{2(0^+)(-3)}\right) \tag{3}$$

$$\lim_{x \to 2^+} \frac{x^3 + 5x^2 + 6x}{2x^3 - 2x^2 - 4x} = \lim_{x \to 2^+} \frac{(x+2)(x+3)}{2(x+1)(x-2)} = \infty \quad \left(\frac{(4)(5)}{2(3)(0^+)}\right) \tag{4}$$

Eqs. (1) and (2) imply that $x = 0$ is not an asymptote (the limit is finite). On the contrary, since the limits in Eqs. (3) and (4) are infinite, $x = -1$ and $x = 2$ are vertical asymptotes. Note: the left hand limits at -1 and/or 2 could also have been computed instead.

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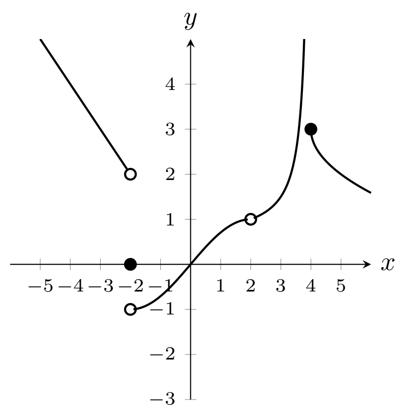
3. [16 pts] Using the graph of $f(x)$ in the figure below, compute the following:

a. $\lim_{x \to -2^-} f(x)$ b. $\lim_{x \to -2^+} f(x)$ c. $\lim_{x \to -2} f(x)$ d. $\lim_{x \to 2} f(x)$

e. $f(2)$ f. $\lim_{x \to 4^-} f(x)$ g. $\lim_{x \to 4+} f(x)$ h. $\lim_{x \to 4} f(x)$



SOLUTION:
a. 2 b. -1 c. Does not exist d. 1 e. Not defined f. ∞ g. 3 h. Does not exist ■

4. (a) [6 pts] What three conditions must be met for a function $f(x)$ to be continuous at the point a?

SOLUTION:
   i. $f(a)$ must be defined.
   ii. $\lim_{x \to a} f(x)$ must exist.
   iii. $\lim_{x \to a} f(x) = f(a)$

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(b) [18 pts] Determine where the following functions are continuous, writing your answer using interval notation.

   i. $f(x) = \sin(\cos(\sin x)) - \cos(\sin(\cos x))$
   ii. $f(x) = \dfrac{x+2}{|x+2|}$
   iii. $f(x) = \begin{cases} \dfrac{\cos 5x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$

SOLUTION:
   i. $(-\infty, \infty)$; Composition/difference of continuous functions in continuous
   ii. $(-\infty, -2) \cup (-2, \infty)$; $f(x) = \begin{cases} 1 & x > -2 \\ -1 & x < -2 \end{cases}$. Jump discontinuity at $x = 2$.
   iii. $(\infty, 0) \cup (0, \infty)$; For $x \neq 0$ the function is the ratio of two continuous functions and is therefore continuous. At $x = 0$, f is defined, $f(0) = 1$, but $\lim_{x \to 0} \dfrac{\cos 5x}{x}$ does not exist.

5. [12 pts] Find the following limits.

(a) $\lim_{x\to\pi/4} \frac{\sin 2x}{2x}$ (b) $\lim_{x\to-\infty} \cos\frac{1}{x}$

SOLUTION:

(a) $\lim_{x\to\pi/4} \frac{\sin 2x}{2x} = \frac{\sin(2\pi/4)}{2\pi/4} = \frac{\sin(\pi/2)}{\pi/2} = \frac{1}{\pi/2} = \frac{2}{\pi}$

(b) $\lim_{x\to-\infty} \cos\frac{1}{x} = \cos\left(\lim_{x\to-\infty}\frac{1}{x}\right) = \cos 0 = 1$

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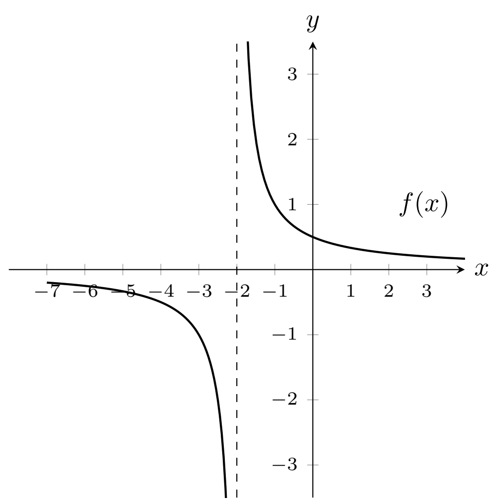
6. Let $f(x) = \frac{1}{x+2}$. If you need to compute any derivatives, you must use the definition.

(a) [5 pts] Find the average rate of change of f over the interval $[3, 8]$. Simplify your answer. What geometric property of the graph of $f(x)$ does this average rate of change represent?

(b) [6 pts] Find the instantaneous rate of change of f at $x = 2$. What geometric property of the graph of $f(x)$ does this instantaneous rate of change represent?

(c) [6 pts] Find the slope/intercept form of the tangent line to the graph of $y = f(x)$ at the point $x = 2$.

(d) [4 pts] The graph of $f(x)$ is shown in the figure below. In your bluebook, sketch a graph of $f'(x)$.



SOLUTION:

(a) Average rate of change of $f = \frac{f(8)-f(3)}{8-3} = \frac{\frac{1}{8+2}-\frac{1}{3+2}}{5} = \frac{\frac{1}{10}-\frac{1}{5}}{5} = -\frac{1}{50}$. This number represents the slope of the secant line between the points $(3, f(3))$ and $(8, f(8))$ on the graph of f.

(b)

$$f'(2) = \lim_{h\to0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to0}\frac{\frac{1}{2+h+2}-\frac{1}{2+2}}{h} = \lim_{h\to0}\frac{\frac{1}{4+h}-\frac{1}{4}}{h} = \lim_{h\to0}\frac{4-(4+h)}{4h(4+h)}$$
$$= \lim_{h\to0}\frac{-h}{4h(4+h)} = \lim_{h\to0}\frac{-1}{4(4+h)} = -\frac{1}{16}$$

Alternatively,

$$f'(2) = \lim_{x\to2}\frac{f(x)-f(2)}{x-2} = \lim_{x\to2}\frac{\frac{1}{x+2}-\frac{1}{4}}{x-2} = \lim_{x\to2}\frac{4-(x+2)}{4(x+2)(x-2)} = \lim_{x\to2}\frac{(2-x)}{4(x+2)(x-2)}$$
$$= \lim_{x\to2}\frac{-1}{4(x+2)} = -\frac{1}{16}$$

This number represents the slope of the line tangent to the graph of f at the point $(2, f(2))$ [or slope of the graph of f at $(2, f(2))$].

(c) $y - f(2) = f'(2)(x-2) \implies y = -\frac{1}{16}(x-2) + \frac{1}{4} \implies y = -\frac{1}{16}x + \frac{1}{8} + \frac{2}{8} \implies y = -\frac{1}{16}x + \frac{3}{8}$

(d) Graph of $f'(x)$