INSTRUCTIONS: Outside paper and electronic devices are **not** permitted. Exam is worth 150 points. Neatness counts. Unless indicated, answers with no supporting work may receive no credit. BOX your final answers.

1. (40 points) No work is required for number 1, parts (a) thru (t). Simply write an answer and box it:

(a) $y = \dfrac{g(x)}{h(x)} \implies \dfrac{dy}{dx} = ?$

(b) $y = g(x) \cdot h(x) \implies \dfrac{dy}{dx} = ?$

(c) $y = [g(x)]^n \implies \dfrac{dy}{dx} = ?$

(d) $y = ax^n \implies \dfrac{dy}{dx} = ?$

(e) $y = \csc(x) \implies \dfrac{dy}{dx} = ?$

(f) $y = \cot(x) \implies \dfrac{dy}{dx} = ?$

(g) $y = \tan(x) \implies \dfrac{dy}{dx} = ?$

(h) $f(y) = x \implies \dfrac{dy}{dx} = ?$

(i) What is the volume of a cone?

(j) What is the volume of a sphere?

(k) What is the surface area of a sphere?

(l) Where are the critical points of $f(x)$ found?

(m) $\sqrt{x^2} = ?$

(n) $\lim_{x \to 0} \left[\dfrac{\sin x}{x} \right] = ?$

(o) What is the limit definition of derivative for $f(x)$?

(p) What must be true of $f(x)$ in order to apply the Mean Value Theorem?

(q) What must be true of $f(x)$ in order to apply the Intermediate Value Theorem?

(r) What is the distance between points (x_1, y_1) and (x_2, y_2)?

(s) If $S(t)$ is the position of an object at time t, then what is $S''(t)$?

(t) When is $f(x)$ considered to be continuous at $x = a$?

Solution:

(a) $\dfrac{h(x)g'(x) - g(x)h'(x)}{h^2(x)}$

(b) $g(x)h'(x) + h(x)g'(x)$

(c) $n[g(x)]^{n-1}g'(x)$

(d) nax^{n-1}

(e) $-\csc x \cot x$

(f) $-\csc^2 x$

(g) $\sec^2 x$

(h) $\dfrac{1}{f'(x)}$

(i) $\dfrac{1}{3}\pi r^2 h$

(j) $\dfrac{4}{3}\pi r^3$

(k) $4\pi r^2$

(l) Where $f'(x) = 0$ or where $f'(x) = \emptyset$

(m) $|x|$

(n) 1

(o) $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

(p) $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b)

(q) $f(x)$ continuous on $[a, b]$, $f(a) < N < f(b)$ and $f(a) \neq f(b)$

(r) $d = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$

(s) Acceleration

(t) $f(a) = \lim_{x \to a} f(x)$

2. (35 points) Consider the function: $f(x) = \sqrt{2x+1}$

(a) Find the average rate of change of $f(x)$ between $x = 4$ and $x = \dfrac{15}{2}$.

(b) Find the instantaneous rate of change of $f(x)$ at $x = 4$.

(c) Create the linearization of $f(x)$ at $a = 4$.

(d) Use the linearization you created in (c) to estimate $\sqrt{10}$.

(e) Is the estimation found in (d) a little too big or a little too small?

(f) We know that $\sqrt{10} > 3$. What is the estimated difference?

(g) We know that $\sqrt{10} > 3$. What is the true difference? Answer with simplified radicals if needed.

Solution: (a) $\dfrac{f(\frac{15}{2}) - f(4)}{\frac{15}{2} - 4} = \dfrac{1}{\frac{7}{2}} = \boxed{\dfrac{2}{7}}$

(b) $f'(x) = \dfrac{1}{\sqrt{2x+1}} \implies f'(4) = \boxed{\dfrac{1}{3}}$

(c) Point $(4, 3)$, slope $\frac{1}{3} \implies y - 3 = \frac{1}{3}(x - 4) \implies \boxed{y = \frac{1}{3}x + \frac{5}{3}}$

(d) $\sqrt{10} = f(4.5) \approx \dfrac{1}{3}\dfrac{9}{2} + \dfrac{5}{3} = \dfrac{9}{6} + \dfrac{10}{6} = \boxed{\dfrac{19}{6}}$

(e) Too big

(f) $dy = \dfrac{dx}{\sqrt{2x+1}} \implies dy = \dfrac{0.5}{3} = \boxed{\dfrac{1}{6}}$

(g) $\boxed{\Delta y = \sqrt{10} - 3}$

3. (20 points) Consider the function: $g(x) = \dfrac{x^4 - 2x^3 + 10x^2 - 20x}{x^2 - 2x}$

(a) Simplify and reduce $g(x)$

(b) Find the absolute minimum and maximum of $g(x)$ on the interval $[3, 5]$.

(c) $\lim_{x\to\infty} \left[\dfrac{1}{g(x)}\right] = ?$

(d) Find any horizontal asymptotes of $h(x) = \dfrac{x^2}{g(x)} + 2$

Solution:

(a) $g(x) = \dfrac{x^3(x-2) + 10x(x-2)}{x(x-2)} = \boxed{x^2 + 10}$

(b) $g(x)$ is continuous on the closed interval $[3, 5]$. $g'(x)$ is defined everywhere, $g'(x) = 0$ when $x = 0$. $g(3) = 19$ and $g(5) = 35$ and $g(0) = \emptyset$, therefore, $\boxed{\text{19 is the minimum}}$, and $\boxed{\text{35 is the maximum}}$.

(c) $\lim_{x\to\infty} \left[\dfrac{1}{x^2+10}\right] = \lim_{x\to\infty} \left[\dfrac{\frac{1}{x^2}}{1 + \frac{10}{x^2}}\right] = \dfrac{0}{1} = \boxed{0}$

(d) $\lim_{x\to\infty} \left[\dfrac{x^2}{x^2+10} + 2\right] = \lim_{x\to\infty} \left[\dfrac{3x^2 + 20}{x^2 + 10}\right] = \lim_{x\to\infty} \left[\dfrac{3 + \frac{20}{x^2}}{1 + \frac{10}{x^2}}\right] = 3 \implies \boxed{y = 3}$ is a HA.

$\lim_{x\to-\infty} \left[\dfrac{3 + \frac{20}{x^2}}{1 + \frac{10}{x^2}}\right] = 3 \implies \boxed{y = 3}$ is a HA., as seen before.

4. (20 points) Consider the function $f(\theta) = \dfrac{\theta^2 - \theta + \theta\cos^2(\theta)}{-\theta^2}$.

(a) Simplify $f(\theta)$ and write your simplification in terms of $\sin(\theta)$.

(b) $\lim_{\theta \to 0}[f(\theta)] =$?

(c) $f'\left(\frac{\pi}{2}\right) =$?

(d) Name two critical points of $f(\theta)$ found in the interval $[0, \pi]$.

Solution:

(a) $f(\theta) = \dfrac{\theta(\cos^2\theta - 1 + \theta)}{-\theta^2} = \boxed{\dfrac{\sin^2\theta - \theta}{\theta}}$

(b) $\lim_{\theta \to 0}\left[\dfrac{\sin\theta}{\theta}\sin\theta - 1\right] = 1 \cdot 0 - 1 = \boxed{-1}$

(c) $f'(\theta) = \dfrac{\theta 2\sin\theta\cos\theta - \sin^2\theta}{\theta^2} = \dfrac{\theta\sin(2\theta) - \sin^2\theta}{\theta^2}$

(d) $f'(\theta) = \dfrac{\theta 2\sin\theta\cos\theta - \sin^2\theta}{\theta^2} = \dfrac{\sin\theta(2\theta\cos\theta - \sin\theta)}{\theta^2} = 0$ or is undefined when $\boxed{\theta = 0 \text{ and when } \theta = \pi}$

5. (20 points) Consider the function: $f(x) = x^2 - 4x + 7$

(a) Use the Intermediate Value Theorem to show that $f(x) = \pi$ for $2 \leq x \leq 6$.

(b) According to the Mean Value Theorem the average rate of change of $f(x)$ between $x = 2$ and $x = 6$ should equal the instantaneous rate of change of $f(x)$ for an x-value inside the interval $[2, 6]$. Where does this occur?

(c) Name the C-value guaranteed to exist by Rolle's Theorem for $f(x)$ on the interval $[1, 3]$.

(d) Name any vertical asymptotes or holes for $g(x) = \dfrac{4x - 4}{f(x) - 4}$

Solution:

(a) $f(x)$ is continuous on $[2, 6]$. $f(2) = 3 < \pi < 19 = f(6)$

(b) $\dfrac{f(6) - f(2)}{6 - 2} = 4 = 2x - 4 = f'(x)$ or $2x - 4 = 4 \implies 2x = 8 \implies \boxed{x = 4}$

(c) $f(x)$ is continuous on $[1, 3]$ and differentiable on $(1, 3)$, and $f(1) = 4 = f(3)$, therefore $f'(x) = 2x - 4 = 0$ for $\boxed{x = 2}$.

(d) $\dfrac{4(x-1)}{x^2 - 4x + 3} = \dfrac{4(x-1)}{(x-3)(x-1)} = \dfrac{4}{x-3} \implies \boxed{x = 3 \text{ is a VA and } (1, -2) \text{ is a hole}}$.

6. (15 points) The following problems are not neccessarily related:

(a) Sketch $f(x) = \sqrt{(x-2)^2}$

(b) Suppose $x(x+2) \leq g(x) + x^2 \leq x^4 + 2$, then $\lim_{x \to 1} [g(x)] = ?$

(c) If the radius of a sphere is increasing at 1/8 millimeters per second, then how fast is the surface area changing when the diameter is 80 mm?

Solution:

(a) a

(b) $x^2 + 2x - x^2 \leq g(x) \leq x^4 - x^2 + 2$

$\lim_{x \to 1} [2x] = 2 = \lim_{x \to 1} [x^4 - x^2 + 2] \implies \boxed{\lim_{x \to 1} [g(x)] = 2}$ by the squeeze theorem.

(c) $S = 4\pi r^2 \implies \frac{ds}{dt} = 8\pi r \frac{dr}{dt}$

$$\left.\frac{ds}{dt}\right|_{r=40} = 8 \cdot \pi \cdot 40 \cdot \frac{1}{8} = \boxed{40 \cdot \pi}$$