INSTRUCTIONS: Books, notes, and electronic devices are not permitted. This exam is worth 100 points. Box your final answers. Write neatly, top to bottom, left to right, one problem per page. A correct answer with incorrect or no supporting work may receive no credit. If you need to find a derivative then you must find it via definition. SHOW ALL WORK

1. (20 points) Evaluate the following limits:

(a) \( \lim_{\theta \to 0} \left[ \frac{\theta \cos^2(\theta) + 10\theta^2 - \theta}{\theta^2} \right] \)

(b) \( \lim_{x \to 2} \left[ \frac{\sqrt{4x + 1} - 3}{x - 2} \right] \)

(c) \( \lim_{t \to 0} \left[ \frac{\tan(6t)}{\sin(2t)} \right] \)

(d) \( \lim_{x \to 2^-} \left[ \frac{x^2 + x - 6}{|x - 2|} \right] \)

Solution:

(a) \( \lim_{\theta \to 0} \left[ \frac{\cos^2 \theta - 1 + 10\theta}{\theta} \right] = \lim_{\theta \to 0} \left[ -1 \cdot \frac{\sin^2 \theta}{\theta} + 10 \right] = \lim_{\theta \to 0} \left[ -1 \cdot \frac{\sin \theta}{\theta} \cdot \sin(\theta) + 10 \right] = 10 \)

(b) \( \lim_{x \to 2} \left[ \frac{\sqrt{4x + 1} - 3 \cdot \sqrt{4x + 1} + 3}{x - 2} \right] = \lim_{x \to 2} \left[ \frac{4x + 1 - 9}{(x - 2) \sqrt{4x + 1} + 3} \right] = \lim_{x \to 2} \left[ \frac{4(x - 2)}{(x - 2) \sqrt{4x + 1} + 3} \right] = \frac{2}{3} \)

(c) \( \lim_{t \to 0} \left[ \frac{\sin(6t)}{\cos(6t) \sin(2t)} \right] = \lim_{t \to 0} \left[ \frac{\sin(2t) \cos(4t) + \sin(4t) \cos(2t)}{\cos(6t) \sin(2t)} \right] = \lim_{t \to 0} \left[ \frac{\cos(4t) + 2 \sin(2t) \cos^2(2t)}{\cos(6t) \sin(2t)} \right] = \frac{1}{1} + 2 \cdot \frac{1^2}{1} = 3 \)

(d) note: \( \frac{x^2 + x - 6}{|x - 2|} = \begin{cases} 
\frac{(x - 2)(x + 3)}{x - 2} = (x + 3) & , x > 2 \\
\frac{(x - 2)(x + 3)}{-(x - 2)} = -(x + 3) & , x < 2 
\end{cases} \)

Therefore, \( \lim_{x \to 2^-} = [-x - 3] = -5 \)
2. (8 points) Consider the function:  

\[ f(x) = \begin{cases} 
-x^2 + 6x - 8 & , x > 3 \\
 x - 2 & , x < 3 \\
 1 & , x = 3 
\end{cases} \]

Show that \( f(x) \) is either continuous on the real numbers, or name any points of discontinuity.

**Solution:**

\( f(x) \) is described in 3 pieces, each of which is a continuous polynomial. Therefore, the only possible point of discontinuity might be at \( x = 3 \).

To demonstrate continuity at \( x = 3 \), we must show \( \lim_{x \to 3} [f(x)] = f(3) \).

i.e. We need \( \lim_{x \to 3^-} [f(x)] = \lim_{x \to 3^+} [f(x)] = f(3) \).

\[ \lim_{x \to 3^-} [x - 2] = 1 \text{ and } \lim_{x \to 3^+} [-x^2 + 6x - 8] = 1 \] and \( f(3) = 1 \).

Therefore, \( f(x) \) is continuous on \((-\infty, \infty)\).
3. (15 points) Consider the equation: \( y = \sqrt{x} \).

(a) Find the average rate of change of \( y \) between \( x = 1 \) and \( x = 4 \).

(b) Find the instantaneous rate of change of \( y \) at \( x = 1 \).

(c) Find the equation of the tangent line to the curve \( y \) at \( x = 1 \).

Solution:

(a) \[
\frac{f(4) - f(1)}{4 - 1} = \frac{\sqrt{4} - \sqrt{1}}{4 - 1} = \frac{2 - 1}{4 - 1} = \frac{1}{3}
\]

(b) \[
\lim_{h \to 0} \left[ \frac{\sqrt{1 + h} - 1}{h} \cdot \frac{\sqrt{1 + h} + 1}{\sqrt{1 + h} + 1} \right] = \lim_{h \to 0} \left[ \frac{1 + h - 1}{h(\sqrt{1 + h} + 1)} \right] = \frac{1}{2}
\]

(c) Using point \((1, 1)\) and \( m = \frac{1}{2} \), we get \( y - 1 = \frac{1}{2} (x - 1) \), or \( y = \frac{1}{2} x + \frac{1}{2} \), or \( x - 2y = -1 \).
4. (12 points) The following may not be related:

(a) \( f(x) = \frac{2}{x} \), then \( f'(1) = ? \)  
(b) \( \lim_{x \to \infty} \left[ \frac{x(2x^2 - 8)}{x^3 - 2x^2 + 100x - 200} \right] = ? \)

**Solution:**

(a) \( f'(x) = \lim_{h \to 0} \left[ \frac{\frac{2}{1+h} - 2}{h} \right] = \lim_{h \to 0} \left[ \frac{2 - 2(1 + h)}{h(1 + h)} \right] = \lim_{h \to 0} \left[ \frac{2 - 2 - 2h}{h(1 + h)} \right] = \lim_{h \to 0} \left[ \frac{-2}{1 + h} \right] = -2 \)

(b) \( \lim_{x \to \infty} \left[ \frac{2x(x + 2)(x - 2)}{(x^2 + 100)(x - 2)} \right] = \lim_{x \to \infty} \left[ \frac{2x^2 + 4x}{x^2 + 100} \right] = \lim_{x \to \infty} \left[ \frac{2 + \frac{4}{x}}{1 + \frac{100}{x^2}} \right] = 2 \)
5. (8 points) Find \( \lim_{x \to 0} \left[ \frac{1 - \cos(x)}{x^2} \right] \), given \( \frac{x^2}{24} < \frac{2 - 2 \cos(x) - x^2}{2x^2} \) and \( \frac{x^2}{2} > |1 - \cos(x)| \).

(Full credit is awarded for using the given information)

**Solution:**

\[
1 - \cos(x) < \frac{x^2}{2} \implies \frac{1 - \cos(x)}{x^2} < \frac{1}{2}. \quad \text{Furthermore,}
\]

\[
\frac{x^2}{24} < \frac{2 - 2 \cos(x) - x^2}{2x^2} \implies \frac{x^2}{24} < \frac{2(1 - \cos(x))}{2x^2} - \frac{1}{2} \implies \frac{1}{2} \frac{x^2}{24} < \frac{1 - \cos(x)}{x^2}
\]

Together these 2 inequalities imply \( \frac{1}{2} \frac{x^2}{24} < \frac{1 - \cos(x)}{x^2} < \frac{1}{2} \)

\[
\lim_{x \to 0} \left[ \frac{1 - x^2}{24} \right] = \frac{1}{2} = \lim_{x \to 0} \left[ \frac{1}{2} \right]
\]

Therefore, \( \lim_{x \to 0} \left[ \frac{1 - \cos(x)}{x^2} \right] = \frac{1}{2} \) by the squeeze theorem.
6. (12 points)
(a) Sketch a function with all six of the characteristics listed below.
(b) Create a function with all six of the characteristics listed below.

(i) \( f(1) = 0 \)  
(ii) \( \lim_{x \to 1} [f(x)] = -1 \)  
(iii) \( \lim_{x \to 3^+} [f(x)] = \infty \)

(iv) \( \lim_{x \to 3^-} [f(x)] = -\infty \)  
(v) \( \lim_{x \to -\infty} [f(x)] = 2 \)  
(vi) \( \lim_{x \to \infty} [f(x)] = 2 \)

Solution:

(a)

(b) \[
f(x) = \frac{(x - 1)2x}{(x - 1)(x - 3)} = \frac{2x^2 - 2x}{x^2 - 4x + 3} = \frac{2x}{x - 3}
\]
MORE ON THE BACK
7. (15 points) Explain why the following statements are true or false. Consider a number of ideas in your explanation: graphs, continuity, increasing functions, the IVT etc. Grading on this problem is dependent on neatness, thoroughness and succinctness of explanation.

(a) The following statement is true, explain why:
\[ f(x) = x^2 - 4x + 7 \] equals \( \pi \) somewhere between \( x = 0 \) and \( x = 6 \).

(b) The following statement is false, explain why:
The I.V.T. can be used to show \( f(x) = \frac{x^3 + 8x + 10}{x - 1} \) equals 10 somewhere between \( x = 0 \) and \( x = 2 \).

(c) The following statement is true, explain why:
\[ f(x) = \frac{8x + 10}{4x^2 + x - 5} \] equals 3 somewhere between \( x = 0 \) and \( x = 2 \).

Solution:
(a) This is an upward opening parabola with vertex \((2, 3)\), and quadratic equations (polynomials) are continuous. Thus all values greater than 3, \( \pi \) included, must be realized. Furthermore, \([0, 2]\) is part of the set of values inside \([0, 6]\). \( f(2) = 3 \), and \( f(0) = 7 \). Since \( 3 < \pi < 7 \), by the IVT we will have \( f(x) = \pi \) for some value between 0 and 2 (and therefore between 0 and 6.)

(b) The IVT cannot be used to determine a root exists between \( x = 0 \) and \( x = 2 \) because \( f(x) \) is not continuous on this interval. There is a discontinuity at \( x = 1 \) and 1 is inside the interval \([0, 2]\). Furthermore, \( f(x) \) is positive only for values \((1, 2]\), inside the interval of concern, and for these values \( f(x) \) is decreasing asymptotically from \(-\infty\) to \( y = 34 \); hence never reaching 10.

(c) \[ f(x) = \frac{2(4x + 5)}{(4x + 5)(x - 1)} = \frac{2}{x - 1} \] This function has a vertical asymptote at \( x = 1 \). \( \lim_{x \to 1^+} f(x) = \infty \) and \( \lim_{x \to \infty} f(x) = 0 \) means that \( f(x) \) will take on all values on \((0, \infty)\), 3 included. Furthermore, \( f\left(\frac{3}{2}\right) = 4 \) and \( f(2) = 2 \). Therefore, \( f(x) \) (being continuous on its domain), by the IVT \( f(x) \) takes on all values between \( y = 2 \) and \( y = 4 \), 3 included.
8. (10 points) Given the following graphs of $f(x)$ and $g(x)$, sketch the graphs of $f'(x)$ and $g'(x)$.
No explanation required. You can use the axis system provided as scratch paper, but you must reproduce a sketch of your graphs in your blue book for any credit.

Solution: