INSTRUCTIONS: Books, notes, and electronic devices are **not** permitted. This exam is worth 100 points. Box your final answers. Write neatly, top to bottom, left to right, one problem per page. A correct answer with incorrect or no supporting work may receive no credit. If you need to find a derivative then you must find it via definition. **SHOW ALL WORK**

1. (20 points) Evaluate the following limits:

(a) $\lim_{\theta\to 0}\left[\frac{\theta\cos^2(\theta)+10\theta^2-\theta}{\theta^2}\right]$ (b) $\lim_{x\to 2}\left[\frac{\sqrt{4x+1}-3}{x-2}\right]$ (c) $\lim_{t\to 0}\left[\frac{\tan(6t)}{\sin(2t)}\right]$ (d) $\lim_{x\to 2^-}\left[\frac{x^2+x-6}{|x-2|}\right]$

Solution:

(a) $\lim_{\theta\to 0}\left[\frac{\cos^2\theta-1+10\theta}{\theta}\right]=\lim_{\theta\to 0}\left[-1\cdot\frac{\sin^2\theta}{\theta}+10\right]=\lim_{\theta\to 0}\left[-1\cdot\frac{\sin\theta}{\theta}\cdot\sin(\theta)+10\right]=\boxed{10}$

(b) $\lim_{x\to 2}\left[\frac{\sqrt{4x+1}-3}{x-2}\cdot\frac{\sqrt{4x+1}+3}{\sqrt{4x+1}+3}\right]=\lim_{x\to 2}\left[\frac{4x+1-9}{(x-2)\left[\sqrt{4x+1}+3\right]}\right]=\lim_{x\to 2}\left[\frac{4(x-2)}{(x-2)\left[\sqrt{4x+1}+3\right]}\right]=$

$\lim_{x\to 2}\left[\frac{4}{\sqrt{4x+1}+3}\right]=\boxed{\frac{2}{3}}$

(c) $\lim_{t\to 0}\left[\frac{\sin(6t)}{\cos(6t)\sin(2t)}\right]=\lim_{t\to 0}\left[\frac{\sin(2t)\cos(4t)+\sin(4t)\cos(2t)}{\cos(6t)\sin(2t)}\right]=\lim_{t\to 0}\left[\frac{\cos(4t)}{\cos(6t)}+\frac{2\sin(2t)\cos^2(2t)}{\cos(6t)\sin(2t)}\right]=$

$\frac{1}{1}+\frac{2\cdot 1^2}{1}=\boxed{3}$

(d) note: $\frac{x^2+x-6}{|x-2|}=\begin{cases}\frac{(x-2)(x+3)}{(x-2)}=(x+3) & ,x>2\\ \frac{(x-2)(x+3)}{-(x-2)}=-(x+3) & ,x<2\end{cases}$

Therefore, $\lim_{x\to 2^-}=[-x-3]=\boxed{-5}$

2. (8 points) Consider the function: $f(x) = \begin{cases} -x^2 + 6x - 8 & , x > 3 \\ x - 2 & , x < 3 \\ 1 & , x = 3 \end{cases}$

Show that $f(x)$ is either continuous on the real numbers, or name any points of discontinuity.

Solution:

$f(x)$ is described in 3 pieces, each of which is a continuous polynomial. Therefore, the only possible point of discontinuity might be at $x = 3$.

To demonstrate continuity at $x = 3$, we must show $\lim_{x\to 3}[f(x)] = f(3)$.

i.e. We need $\lim_{x\to 3^-}[f(x)] = \lim_{x\to 3^+}[f(x)] = f(3)$.

$\lim_{x\to 3^-}[x - 2] = 1$ and

$\lim_{x\to 3^+}[-x^2 + 6x - 8] = 1$ and

$f(3) = 1$.

Therefore, $f(x)$ is continuous on $(-\infty, \infty)$.

3. (15 points) Consider the equation: $y = \sqrt{x}$.

(a) Find the average rate of change of y between $x = 1$ and $x = 4$.

(b) Find the instantaneous rate of change of y at $x = 1$.

(c) Find the equation of the tangent line to the curve y at $x = 1$.

Solution:

(a) $\dfrac{f(4) - f(1)}{4 - 1} = \dfrac{\sqrt{4} - \sqrt{1}}{4 - 1} = \dfrac{2 - 1}{4 - 1} = \boxed{\dfrac{1}{3}}$

(b) $\lim_{h \to 0} \left[\dfrac{\sqrt{1+h} - 1}{h} \cdot \dfrac{\sqrt{1+h} + 1}{\sqrt{1+h} + 1} \right] = \lim_{h \to 0} \left[\dfrac{1 + h - 1}{h(\sqrt{1+h} + 1)} \right] = \boxed{\dfrac{1}{2}}$

(c) Using point $(1, 1)$ and $m = \dfrac{1}{2}$, we get $\boxed{y - 1 = \dfrac{1}{2}(x - 1)}$, or $\boxed{y = \dfrac{1}{2}x + \dfrac{1}{2}}$, or $\boxed{x - 2y = -1}$.

4. (12 points) The following may not be related:

(a) $f(x) = \frac{2}{x}$, then $f'(1) = ?$ (b) $\lim_{x\to\infty} \left[\frac{x(2x^2-8)}{x^3-2x^2+100x-200} \right] = ?$

Solution:

(a) $f'(x) = \lim_{h\to 0} \left[\frac{\frac{2}{1+h} - 2}{h} \right] = \lim_{h\to 0} \left[\frac{2-2(1+h)}{h(1+h)} \right] = \lim_{h\to 0} \left[\frac{2-2-2h}{h(1+h)} \right] = \lim_{h\to 0} \left[\frac{-2}{1+h} \right] = \boxed{-2}$

(b) $\lim_{x\to\infty} \left[\frac{2x(x+2)(x-2)}{(x^2+100)(x-2)} \right] = \lim_{x\to\infty} \left[\frac{2x^2+4x}{x^2+100} \right] = \lim_{x\to\infty} \left[\frac{2+\frac{4}{x}}{1+\frac{100}{x^2}} \right] = \boxed{2}$

5. (8 points) Find $\lim_{x\to 0}\left[\frac{1-\cos(x)}{x^2}\right]$, given $-\frac{x^2}{24} < \frac{2-2\cos(x)-x^2}{2x^2}$ and $\frac{x^2}{2} > [1-\cos(x)]$.
(Full credit is awarded for using the given information)

Solution:

$1-\cos(x) < \frac{x^2}{2} \implies \frac{1-\cos(x)}{x^2} < \frac{1}{2}$. Furthermore,

$$-\frac{x^2}{24} < \frac{2-2\cos(x)-x^2}{2x^2} \implies -\frac{x^2}{24} < \frac{2(1-\cos(x))}{2x^2} - \frac{1}{2} \implies \frac{1}{2} - \frac{x^2}{24} < \frac{1-\cos(x)}{x^2}$$

together these 2 inequalities imply $\frac{1}{2} - \frac{x^2}{24} < \frac{1-\cos(x)}{x^2} < \frac{1}{2}$

$$\lim_{x\to 0}\left[\frac{1}{2} - \frac{x^2}{24}\right] = \frac{1}{2} = \lim_{x\to 0}\left[\frac{1}{2}\right]$$

Therefore, $\lim_{x\to 0}\left[\frac{1-\cos(x)}{x^2}\right] = \boxed{\frac{1}{2}}$ by the squeeze theorem.

6. (12 points)

(a) Sketch a function with all six of the characteristics listed below.

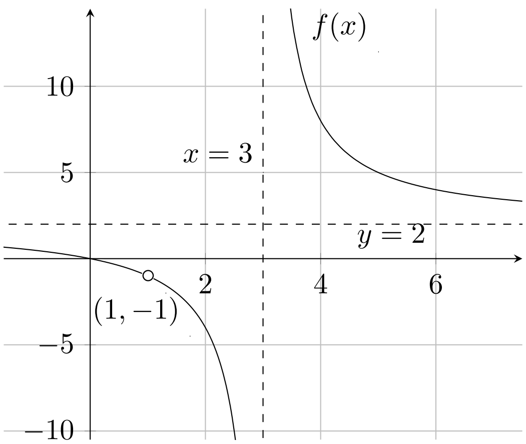
(b) Create a function with all six of the characteristics listed below.

(i) $f(1) = \emptyset$ (ii) $\lim_{x\to 1}[f(x)] = -1$ (iii) $\lim_{x\to 3^+}[f(x)] = \infty$

(iv) $\lim_{x\to 3^-}[f(x)] = -\infty$ (v) $\lim_{x\to -\infty}[f(x)] = 2$ (vi) $\lim_{x\to \infty}[f(x)] = 2$

Solution:

(a)



(b) $\boxed{f(x) = \dfrac{(x-1)2x}{(x-1)(x-3)} = \dfrac{2x^2-2x}{x^2-4x+3} = \dfrac{2x}{(x-3)}}$

MORE ON THE BACK

7. (15 points) Explain why the following statements are true or false.
Consider a number of ideas in your explanation: graphs, continuity, increasing functions, the IVT etc.
Grading on this problem is dependent on neatness, thoroughness and succinctness of explanation.

(a) The following statement is true, explain why:
$f(x) = x^2 - 4x + 7$ equals π somewhere between $x = 0$ and $x = 6$.

(b) The following statement is false, explain why:
The I.V.T. can be used to show $f(x) = \dfrac{x^3 + 8x + 10}{x - 1}$ equals 10 somewhere between $x = 0$ and $x = 2$.

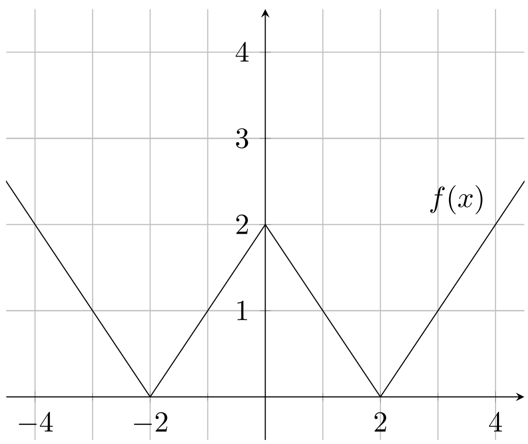
(c) The following statement is true, explain why:
$f(x) = \dfrac{8x + 10}{4x^2 + x - 5}$ equals 3 somewhere between $x = 0$ and $x = 2$.

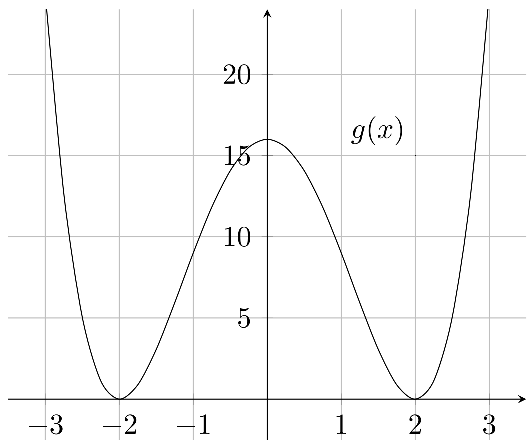
Solution:
(a) This is an upward opening parabola with vertex $(2, 3)$, and quadratic equations (polynomials) are continuous. Thus all values greater than 3, π included, must be realized.
Furthermore, $[0, 2]$ is part of the set of values inside $[0, 6]$. $f(2) = 3$, and $f(0) = 7$. Since $3 < \pi < 7$, by the IVT we will have $f(x) = \pi$ for some value between 0 and 2 (and therefore between 0 and 6.)

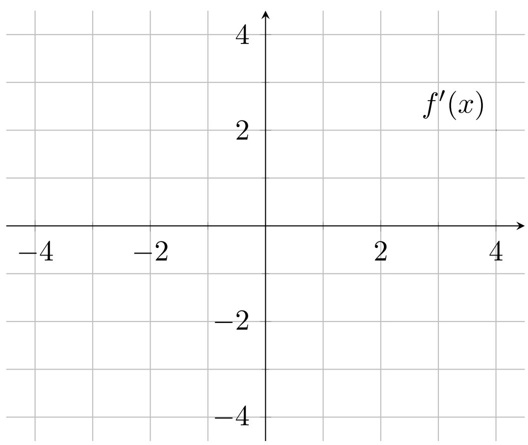
(b) The IVT cannot be used to determine a root exists bewteen $x = 0$ and $x = 2$ because $f(x)$ is not continuous on this interval. There is a discontinuity at $x = 1$ and 1 is inside the interval $[0, 2]$.
Furthermore, $f(x)$ is positive only for values $(1, 2]$, inside the interval of concern, and for these values $f(x)$ is decreasing asymptotically from $-\infty$ to $y = 34$; hence never reaching 10.

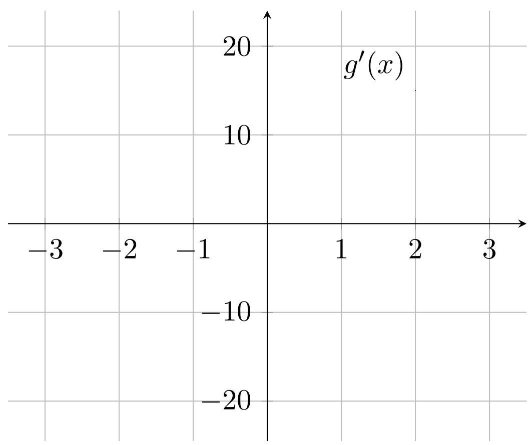
(c) $f(x) = \dfrac{2(4x + 5)}{(4x + 5)(x - 1)} = \dfrac{2}{x - 1}$. This function has a vertical asymptote at $x = 1$. $\lim_{x \to 1^+} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = 0$ means that $f(x)$ will take on all values on $(0, \infty)$, 3 included.
Furthermore, $f\left(\frac{3}{2}\right) = 4$ and $f(2) = 2$. Therefore, $f(x)$ (being continuous on its domain), by the IVT $f(x)$ takes on all values between $y = 2$ and $y = 4$, 3 included.

8. (10 points) Given the following graphs of $f(x)$ and $g(x)$, sketch the graphs of $f'(x)$ and $g'(x)$. No explanation required. You can use the axis system provided as scratch paper, but you must reproduce a sketch of your graphs in your blue book for any credit.









Solution: