INSTRUCTIONS: Books, notes, and electronic devices are not permitted. This exam is worth 150 points. Box your final answers. Write neatly, top to bottom, left to right, one problem per page. A correct answer with incorrect or no supporting work may receive no credit. SHOW ALL WORK

1. (14 points; 6,8)
   (a) Find \( \frac{dy}{dx} \) given that \( y \cos(x) = 1 + \sin(xy) \).

   (b) Suppose \( f(x) = \sec^2(3x) \) and \( g(x) = \tan^2(3x) \) were graphed on the same coordinate system, along with their corresponding tangent lines at \( x = \frac{\pi}{12} \).
   If the tangent lines are parallel then how far apart are their \( y \)-coordinates?
   If the tangent lines are not parallel then what is their point of intersection?

   Solution:
   (a) Implicitly derivating \( y \cos(x) = 1 + \sin(xy) \) produces:

   \[
   y(-\sin x) + \cos(x)y' = \cos(xy) + y \cos(xy) = y'x \cos(xy) + y \cos(xy) \\
   y'(\cos x - x \cos(xy)) = y \cos(xy) + y \sin(x)
   \]

   \[
   y' = \frac{y \cos(xy) + y \sin(x)}{\cos(x) - x \cos(xy)}
   \]

   (b) \( f(x) = \sec^2(3x) \) and \( g(x) = \tan^2(3x) \) means that:

   \( f\left(\frac{\pi}{12}\right) = 2 \) and \( g\left(\frac{\pi}{12}\right) = 1 \). Therefore, we have points of tangency \( \left(\frac{\pi}{12}, 2\right) \) and \( \left(\frac{\pi}{12}, 1\right) \) on \( f(x) \) and \( g(x) \).

   The slopes of the tangent lines are found with \( f'(\frac{\pi}{12}) \) and \( g'(\frac{\pi}{12}) \). Note \( f'(\frac{\pi}{12}) = g'(\frac{\pi}{12}) = 12 \)

   The tangent line equations that we seek are: \( y = 12x + (2 - \pi) \) and \( y = 12x + (1 - \pi) \). These are lines are parallel.

   The difference in their \( y \)-coordinates is \( (2 - \pi) - (1 - \pi) = 1 \).
2. (24 points; 6*4) Consider the functional relationship $f(x) = |(x - 2)^2 - 4|$. 
(a) Sketch a graph of $f(x)$.
(b) What is the domain of $f(x)$?
(c) What is the range of $f(x)$?
(d) What is $f'(x)$?
(e) Sketch $h(x) = -x^2 + k$ for any $k$.
(f) Suppose there is a function $g(x)$ such that $h(x) \leq g(x) \leq f(x)$ for all $x$ for the functions $h(x)$ and $f(x)$ defined above. The Squeeze theorem can be used to find the $\lim_{x \to a} g(x)$ for what values of $a$ and $k$? Justify your answer using appropriate limits.

Solution:

(a) 

(b) Domain is all real numbers.

(c) Range is $[0, \infty)$. 

(d) $f'(x) = \begin{cases} 2(x - 2) & : x < 2, x > 4 \\ -2(x - 2) & : 0 < x < 4 \end{cases}$

(e) Consider $h(x) = -x^2 + 0$, i.e. $k = 0$.

(f) For $a = 0$ and $k = 0$ we have $\lim_{x \to 0} g(x) = 0$ by the squeeze theorem.
3. (16 points; 6,10) 
(a) State the definition of continuity of a function at $x = a$.

(b) Using the definition of continuity and the function below, show whether or not $f(x)$ is continuous at $x = 0$.

$$f(x) = \begin{cases} 
\frac{3x-x^2}{5x-x^2} & : x < 0 \\
\frac{3}{5} & : x = 0 \\
tan(3x) \csc(5x) & : x > 0
\end{cases}$$

Solution:

(a) $f(x)$ is continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$.

(b) $\lim_{x \to 0^-} \left[ \frac{3x-x^2}{5x-x^2} \right] = \lim_{x \to 0^-} \left[ \frac{3-x}{5-x} \right] = \frac{3}{5}$.

$\lim_{x \to 0^+} \tan(3x) \csc(5x) = \lim_{x \to 0^+} \left[ \frac{\sin(3x)}{\cos(3x)} \cdot \frac{1}{\csc(5x)} \cdot \frac{3x}{3x} \cdot \frac{5x}{5x} \right]$

$= \lim_{x \to 0^+} \left[ \frac{\sin(3x)}{\sin(5x)} \cdot \frac{5x}{3x} \cdot \frac{1}{\cos(3x)} \cdot \frac{5x}{5x} \right]$

$= 1 \cdot 1 \cdot 1 \cdot \frac{3}{5} = \frac{3}{5}$

Since $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = \frac{3}{5}$ we can say $\lim_{x \to 0} f(x) = \frac{3}{5}$.

Furthermore, $f(0) = \frac{3}{5}$, therefore $f(x)$ is continuous at $x = 0$. 
4. (22 points; 6,6,10)
(a) Find the linearization of \( \sec(\theta) \) at \( \theta = \pi \).

(b) Use the linearization to estimate the value of \( \sec \left( \frac{9\pi}{10} \right) \).

(c) The radius of a sphere was measured and found to be 10 cm with a possible error in measurement of at most 0.02 cm. Using differentials, find the maximum error in using this value of the radius to compute the volume of the sphere and report your solution as a percentage error.

Solution:
(a) \( y = \sec(x) \implies y(\pi) = \pi \) and \( y' = \sec(x) \tan(x) \) and \( y'(\pi) = 0 \) therefore our tangent line is \( y = -1 \).

(b) \( \sec \left( \frac{9\pi}{10} \right) \approx \boxed{y = -1} \)

(c) \( r = 10, \ dr = 0.02, \ v = \frac{4}{3} \pi r^3, \ dv = 4\pi r^2 dr = 4\pi 10^2 (0.02) = \frac{800\pi}{100} = 8\pi \)

percentage error = \( \frac{dv}{v} = \frac{4\pi r^2 dr}{\frac{4}{3} \pi r^3} = \frac{3dr}{r} = \frac{3(0.02)}{10} = \frac{6}{1000} = 0.006 \) or \( 0.6\% \).
5. (14 points) Consider the function \( g(x) = x^{\frac{2}{3}} + x \) on the interval \([-1, 0]\).
If \( g(x) \) satisfies Rolle’s Theorem then find the \( C \) value that the theorem indicates must exist.
If \( g(x) \) does not satisfy Rolle’s Theorem then state the condition that fails to be met.

**Solution:** \( g(x) \) is a continuous function on the closed interval \([-1, 0]\), it is differentiable on the interval \((-1, 0)\), and \( g(-1) = g(0) = 0 \).
According to Rolle’s Theorem there exists a \( C \) such that \( g'(C) = \frac{2}{3} C^{-\frac{1}{3}} + 1 = 0 \implies 2 = -3\sqrt[3]{C} \implies C = \frac{8}{27} \).
6. (16 points) Consider the function \( g(x) = V(x) - f(x) \) where \( V(x) \) is the volume of the solid below (All corners are 90° angles) and \( f(x) = x^3 + 14x^2 - 8x + 10 \).
If \( g(3) < g(2) < g(4) \) then demonstrate that \( g(3) \) is both a local and an absolute minimum of \( g(x) \) on the interval \([2, 4]\).

**Solution:** \( V(x) = (x + 1)(x + 2)(2x + 1) = 2x^3 + 7x^2 + 7x + 2 \)

Therefore, \( g(x) = x^3 - 7x^2 + 15x - 8 \) which is continuous on \([2, 4]\).
By the EVT \( g(x) \) has absolute extrema on \([2, 4]\).

\[
g'(x) = 3x^2 - 14x + 15 = (3x - 5)(x - 3) \implies g'(x) = 0 \text{ when } x = \frac{5}{3} \text{ or } x = 3.
\]
\( x = 3 \) is the only critical point found in the interval \([2, 4]\).

\( g(3) < g(2) < g(4) \implies g(3) \) is the absolute minimum, and not being an endpoint means \( g(3) \) is a local minimum as well.
7. (18 points; 6,6,6) Suppose the position of an object at time $t$ is given by $s(t) = -\frac{1}{3}t^3 + 3t^2 - 8t - 4$.

(a) Find the velocity of the object at time $t$.

(b) When is the acceleration negative?

(c) When does the object attain maximum speed?

**Solution:**

(a) Velocity is the derivative of position: $v(t) = -t^2 + 6t - 8$.

(b) Acceleration is the derivative of velocity: $a(t) = -2t + 6$. $a(t) < 0 \implies t > 3$.

(c) Maximum velocity occurs at the vertex of the velocity parabola, i.e. when $t = 3$. 

8. (12 points) Gravel is being dumped from a conveyor belt at a rate of \( \frac{30 \text{ ft}^3}{\text{min}} \) and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 ft high?

Solution:

(a) \( \frac{dv}{dt} = 30 \) and \( h = 2r \implies r = \frac{h}{2} \) and \( v(r, h) = \frac{1}{3} \pi r^2 h \) so \( v(h) = \frac{1}{3} \pi \left( \frac{h}{2} \right)^2 h = \frac{\pi}{12} h^3 \)

\[
\frac{dv}{dt} = \frac{3\pi}{12} h^2 \frac{dh}{dt} = 30
\]

\[
\frac{dh}{dt} \bigg|_{h=10} = \frac{30 \cdot 12}{3\pi 10^2} = \frac{120}{100\pi} = \frac{6}{5\pi}
\]
9. (14 points; 7*2) Sketch a functional relationship \( f(x) \) such that:

(a) \( \lim_{x \to -\infty} f(x) = 0 \)

(b) \( \lim_{x \to \infty} f(x) = \infty \)

(c) \( \lim_{x \to 0} f(x) = \infty \)

(d) \( f(a) = \emptyset \) for a particular value \( x = a > 0 \).

(e) \( \lim_{x \to a} f(x) = 0 \)

(f) \( f(x) \) has no positive roots.

(g) \( f(x) \) has infinitely many negative roots.

**Solution:**

(a)