1. [3 @ 8 points each] Larry the Loan Shark is in the business of making high-interest loans. Loan payments are calculated according to the formula

\[
X = \frac{iP}{1 - (1 + i)^{-N}}
\]

where \( P \) is the amount borrowed in dollars, \( N \) is the number of payments to be made, \( X \) is the amount of each loan payment in dollars, and \( i \) is the interest rate that accrues with every payment period. Assume for the following questions that Larry charges 100% interest, or \( i = 1 \).

(a) Find an equation for the amount borrowed \( P \) (in other words, solve the equation above for \( P \)).

(b) Find an equation for the number of payments \( N \) (in other words, solve the equation above for \( N \)).

(c) Suppose some hapless fellow borrows $100 from Larry and promises to pay the loan back in 2 payments. Use the formula above to determine how much his payments will be.

Solution

(a) \( P = X \left[ 1 - z^{-N} \right] \)

(b) \[
\frac{P}{X} = 1 - z^{-N} \\
z^{-N} = 1 - \frac{P}{X} \\
-N = \log_z \left[ 1 - \frac{P}{X} \right] \\
N = -\log_z \left[ 1 - \frac{P}{X} \right]
\]

(c) \[
X = \frac{100}{1 - 2^{-2}} = \frac{100}{1 - 1/4} = \frac{4}{3} \times 100 \approx $133
\]
2. [3 @ 8 points each] These questions are not related.

(a) Solve for $x$: $(\log_2 x)^2 - 4 \log_2 x + \ln e^3 = 0$.

(b) For $f(x) = e^{2x} - 5$, find $f^{-1}$, then state the domain and range of $f^{-1}$ in interval notation.

(c) Sketch the graph of the function $g(x) = \frac{(x - 4) \log_2 x}{x - 4}$. On your graph:
   - Label any and all $x$-intercepts, $y$-intercepts and holes with their $(x, y)$ coordinates.
   - Label any and all asymptotes with their equations.

Solution

(a) $(\log_2 x)^2 - 4 \log_2 x + 3 = 0$ ($\ln e^3 = 3$)

Let $u = \log_2 x$

$u^2 - 4u + 3 = 0$

$(u - 3)(u - 1) = 0$

$u = 1, 3 \Rightarrow \log_2 x = 1, \log_2 x = 3$

$\therefore x = 2, 8$

(b) $y = e^{2x} - 5$

$x \leftrightarrow y: x = e^{2y} - 5$

$x + 5 = e^{2y}$

$\ln (x + 5) = 2y$

$f^{-1}(x) = \frac{1}{2} \ln (x + 5)$

Domain: $(-5, \infty)$

Range: $(-\infty, \infty)$

(c) Vertical Asymptote: $x = 0$

Hole: $(4, 2)$

$x$-intercept: $(1, 0)$
3. [2 @ 8 points each] These questions are not related.

(a) Find a possible equation for the function \( h(x) \) shown to the right. Give your function in factored form.

(b) Find the equation of a rational function \( k(x) \) that has all of the following:
   • vertical asymptotes at \( x = 4 \) and \( x = -4 \) and no others
   • horizontal asymptote at \( y = 0 \)
   • \( y \)-intercept at \( (0, 2) \)
   • no \( x \)-intercepts

Solution

(a) \( h(x) = x^2(x+3)(x-5) \)

(b) \( k(x) = \frac{ax+b}{(x+4)(x-4)} \)

no \( x \)-intercepts means \( a = 0 \)

\( k(x) = \frac{b}{(x+4)(x-4)} \)

\( y \)-intercept \( @ (0, 2) \)

\( k(0) = \frac{b}{-16} = 2 \Rightarrow b = -32 \)

\( k(x) = \frac{-32}{(x+4)(x-4)} \)
4. [3 @ 6 points each] Archimedes' formula says that the area of between a parabola and the $x$-axis is

\[ A = \frac{2}{3} (\text{base})(\text{height}) \]

where “base” is the distance between the $x$-intercepts and “height” is the height of the vertex above the $x$-axis, as shown in the figure. Answer the following questions to find the area between the parabola $y = -x^2 + 7x - 10$ and the $x$-axis.

(a) Compute the base of the parabola.

(b) Compute the height of the parabola.

(c) Find the area between the parabola and the $x$-axis.

\[ y = -x^2 + 7x - 10 \]

\[ \cdot \cdot \cdot \text{base} = \boxed{3} \]

\[ \cdot \cdot \cdot \text{height} = \boxed{\frac{9}{4}} \]

\[ \left( c \right) \text{Area} = \frac{2}{3} (3)(\frac{9}{4}) = \boxed{\frac{9}{2}} \]
5. [3 @ 6 points each] These questions will be graded on your answers only. You do not need to show work.

Answer questions (a), (b) and (c) concerning the function \( R(x) = \frac{x-2}{x^2} \):

(a) Is the symmetry of \( R(x) \) odd, even or neither? \[ \text{Neither} \]

(b) The function \( R(x) \) has a vertical asymptote at \( x = 0 \). Which of the following figures shows the behavior of the function around the vertical asymptote?

A. \[ \text{Figure A} \]
B. \[ \text{Figure B} \]
C. \[ \text{Figure C} \]
D. \[ \text{Figure D} \]

(c) Identify any horizontal or slant asymptotes of \( R(x) \).

Solution \[ \text{HA: } y = 0 \]