AT THE TOP OF THE PAGE write your name and your section number. Textbooks, class notes and electronic devices of any kind are NOT permitted.

Do all your work on this exam.

Part I. Chapter 7 Show Your Work, 60 points
Part II. Comprehensive Show Your Work, 40 points
Part III. Comprehensive MC/TF/Matching, 50 points
Total: 150 points

Part I. Chapter 7 Show Your Work. Fully simplify all solutions. For these problems you must show a complete and valid solution method for full credit.

1. (a) [15 points] Find the solutions of the equation $\sin 3x = \frac{\sqrt{3}}{2}$ in the interval [0, 2π).

$$3x = \frac{\pi}{3}, \frac{2\pi}{3}$$

$$3x = \frac{\pi}{3} + 2\pi k, \frac{2\pi}{3} + 2\pi k \Rightarrow x = \frac{\pi}{9} + \frac{2\pi}{3} k, \frac{2\pi}{9} + \frac{2\pi}{3} k$$

$k=0$: $\frac{\pi}{9}, \frac{2\pi}{3}

k=1$: $\frac{7\pi}{9}, \frac{8\pi}{3}$

$k=2$: $\frac{13\pi}{9}, \frac{14\pi}{3}$

Answer: __________________________

(b) [15 points] Find all of the solutions of the equation $\tan \theta + 1 = \sec \theta$.

\[
\frac{\sin \theta}{\cos \theta} + 1 = \frac{1}{\cos \theta} \rightarrow \sin \theta + \cos \theta = 1
\]

\[
(\sin \theta + \cos \theta)^2 = 1^2
\]

\[
\sin^2 \theta + 2\sin \theta \cos \theta + \cos^2 \theta = 1
\]

\[
2 \sin \theta \cos \theta = 0
\]

\[
\sin \theta \cos \theta = 0
\]

\[
\sin \theta = 0 \rightarrow \theta = 0, \pi
\]

\[
\cos \theta = 0 \rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}
\]

\[
\text{Check solutions:}
\]

\[
\theta = 0: 0 + 1 = 1 \checkmark
\]

\[
\theta = \pi: 0 + 1 = -1 \times
\]

\[
\theta = \frac{\pi}{2}, \frac{3\pi}{2}: \tan \theta \text{ undefined}
\]

Answer: __________________________

\[
\theta = 2\pi k
\]
2. [15 points] Find the exact value of the expression \( \sin^2\left(\frac{1}{2} \cos^{-1}\left(\frac{3}{5}\right)\right) \).

\[
\cos \theta = \frac{3}{5} \quad \text{[Diagram: triangle with sides 3, 4, 5]}
\]

\[
\sin^2 u = \frac{1-\cos^2 u}{2}
\]

\[
\sin^2\left(\frac{1}{2} \theta\right) = \frac{1-\cos \theta}{2}
\]

\[
= 1 - \frac{3}{5}
\]

\[
= \frac{2}{5}
\]

Answer: \__________________________

3. [15 points] Write \( \sin(3\theta) \) entirely in terms of \( \sin \theta \) and/or \( \sin \theta \) raised to a power.

\[
\sin 3\theta = \sin (2\theta + \theta)
\]

\[
= \sin 2\theta \cos \theta + \cos 2\theta \sin \theta
\]

\[
= (2\sin \theta \cos \theta) \cos \theta + (\cos^2 \theta - \sin^2 \theta) \sin \theta
\]

\[
= 2\sin \theta \cos^2 \theta + \cos^2 \theta \sin \theta - \sin^3 \theta
\]

\[
= 2\sin \theta \cos^2 \theta + \cos \theta \sin \theta - \sin^3 \theta
\]

\[
= 3\sin \theta \cos^2 \theta - \sin^3 \theta
\]

\[
= 3 \sin \theta (1 - \sin^2 \theta) - \sin^3 \theta
\]

\[
= 3 \sin \theta - 3\sin^3 \theta - \sin^3 \theta
\]

\[
= 3 \sin \theta - 4 \sin^3 \theta
\]

Answer: \__________________________
Part II. Comprehensive Show Your Work. Fully simplify all solutions. For these problems you must show a complete and valid solution method for full credit. **Draw a box around your final answer for each question.**

4. [20 points] A five-sided box (an open box, no top) is to be constructed by cutting away the corners of a sheet of cardboard and folding up the sides. The sheet of cardboard is 12” wide by 24” long. A diagram of how the cardboard is to be cut is shown below.

![Diagram of cardboard and box](image)

(a) Find an equation for the volume $V$ of the box as a cubic function of $x$. Call this function $V(x)$. Leave your equation in factored form.

$$V(x) = x(24 - 2x)(12 - 2x) = 4x^3 + \ldots$$

(b) For what values of $x$ is $V(x) \geq 0$?

$$[0, 6], [12, \infty)$$

(c) The maximum volume of the box is roughly halfway between the two smaller $x$-intercepts of the function. Estimate the maximum volume by finding the volume at the value of $x$ that is halfway between the two smaller $x$-intercepts.

$$V(3) = 324 \text{ in}^3$$

(d) Sketch the graph of the function $V(x)$. Label the locations of the $x$-intercepts.
5. [20 points] Let's suppose that at 7 am this morning you consumed a 10-oz cup of coffee that contains 140 mg of caffeine. The equation

\[ C(t) = 140 \left( \frac{1}{2} \right)^{t/6} \]

describes the amount \( C(t) \) in milligrams of caffeine remaining in your system \( t \) hours since 7 am. Answer the following questions assuming that in terms of metabolism you are an average adult.

(a) What is the half-life of the caffeine in your system?

\[ 6 \text{ hrs} \]

(b) How much caffeine will remain in your system by the end of this exam at 10 am? Use the fact that \( \sqrt{2} \approx 1.4 \) to get an approximate value for the answer. [Hint: Your answer should be an integer.]

\[ C = 140 \left( \frac{1}{2} \right)^{3/6} = 140 \cdot \frac{1}{\sqrt{2}} = 140 \cdot \frac{1}{1.4} \]

\[ \sqrt{2} = 1.4 \]

\[ 100 \text{ mg} \]

(c) When will the amount of caffeine in your system be 10\% of the amount you consumed? Use the fact that \( \log \left( \frac{1}{2} \right) \approx -0.30 \) to get an approximate value for the answer. [Hint: Your answer should be an integer.]

\[ \frac{14}{140} = \left( \frac{1}{2} \right)^{t/6} \]

\[ 0.1 = \left( \frac{1}{2} \right)^{t/6} \]

\[ \frac{t}{6} = \log_2(0.1) \approx 0.7 \]

\[ t = 20 \text{ hrs} \]

(d) Solve the \( C(t) \) equation given above to get an equation for \( t \) in terms of \( C \).

\[ t = 6 \log \left( \frac{C}{140} \right) = -20 \log \left( \frac{C}{140} \right) \]
Part III. Comprehensive Multiple Choice, True/False, Matching. [4 points each MC/TF] These problems will be graded on your answers only. There is one and only one correct answer for each question. Remember that for True/False questions, if any part of the statement is false, then the entire statement is false.

6. Which function does NOT have an asymptote at \( x = 0 \)?

A. \( \log_4 x \)
B. \( 5^x \)
C. \( \frac{1}{x^2 - 2x} \)
D. \( \tan(x - \frac{\pi}{2}) \)

7. True or false: A function can never cross a vertical asymptote but may cross a horizontal asymptote. (Circle your answer.)

8. True or false: \( a^{-2x} = \left(\frac{1}{\sqrt{a}}\right)^x \)

9. The graph of \( y = \log_2 (4 - 3x) \) has an x-intercept at

A. \( x = 1 \)
B. \( x = 4 / 3 \)
C. \( x = -4 \)
D. There is no x-intercept.

10. The relation \( \sin^{-1}(\sin x) = x \) is true for \( x \) in which quadrants?

A. I and II
B. I and III
C. I and IV
D. I, II, III and IV

11. Three of the four functions given below produce exactly the same graph. Which function does NOT produce the same graph as the other three?

A. \( y = \sin(2x - \pi) \)
B. \( y = \cos 2(x - \frac{\pi}{4}) \)
C. \( y = \cos(2x + \frac{3\pi}{2}) \)
D. \( y = -\sin 2(x + \frac{\pi}{2}) \)

12. True or false: \( \cos^{-1} x = \frac{1}{\cos x} \).

13. A wheel travels a distance of \( \frac{3\pi}{2} \) feet in one revolution. What is the diameter of the wheel?

A. 3/2 feet
B. 3/8 feet
C. 3/4 feet
D. \( \sqrt{3}/2 \) feet
14. Rewrite the expression \( \tan \left( \cos^{-1} \frac{1}{x} \right) \) as an algebraic expression in \( x \).

A. \( \sqrt{x^2 - 1} \)
B. \( \sqrt{1 + x^2} \)
C. \( \frac{1}{\sqrt{x^2 - 1}} \)
D. \( \frac{\sqrt{1 - x^2}}{x} \)

15. The graph of a polynomial function is shown below. Which one of the following could be that function?

![Graph of a polynomial function](image)

A. \( y = -4x^2 - 3x + 10 \)
B. \( y = x^3 + 4x^2 - 3x - 10 \)
C. \( y = -2x^4 + 3x^3 - x^2 - 20 \)
D. \( y = -x^4 - 4x^2 + x + 10 \)

16. What do the functions \( e^x, \sqrt{x}, x^3, \ln x \) and \( \tan x \) have in common?

A. The value of each function at \( x = 2 \) is irrational.
B. They are all one-to-one functions.
C. They are all strictly increasing functions.
D. Each has an intercept at the origin.

17. [6 points] Match each function to its graph. Write the letter for the function in the box next to its graph. **One graph will not be used.** Asymptotes are shown as dotted lines. The graphs are not necessarily to the same scale.

A. \( f(x) = \frac{1}{x^2 - 4} \)
B. \( f(x) = \frac{x}{x^2 - 4} \)
C. \( f(x) = \frac{-x^2}{x^2 - 4} \)
D. \( f(x) = \frac{1}{x - 2} \)
E. \( f(x) = \frac{2x}{x - 2} \)

END OF EXAM